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Abstract

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SU YU-CHEN

ASYMPTOTICS OF SOLUTIONS OF CERTAIN DEGENERATING QUASILINEAR HYPERBOLIC EQUATIONS OF SECOND ORDER

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In the present note an asymptotic expansion with respect to the small parameter ε is constructed for the solutions of the Cauchy problem and of a mixed problem for a quasilinear partial differential equation of hyperbolic type with small parameter ε, and for a boundary-value problem in the case of an equation of elliptic type. For quasilinear ordinary differential equations with a small parameter, the Cauchy problem and the boundary-value problem were studied in work ([1]). In the present note we use the methods of ([1,2]).

1. **The Cauchy problem.** In the domain $G : \{-\infty < x < +\infty,\ t \geqslant 0\}$ we consider the following problem, which we call problem A_ε:

$$L_\varepsilon u_\varepsilon \equiv \varepsilon \left(\partial^2 u_\varepsilon/\partial x^2 - \partial^2 u_\varepsilon/\partial t^2\right) - \varphi(x,t,u_\varepsilon)\, \partial u_\varepsilon/\partial t + \psi(x,t,u_\varepsilon) = 0; \quad (1)$$

$$u_\varepsilon\big|_{t=0} = \alpha(x), \qquad \partial u_\varepsilon/\partial t\big|_{t=0} = \beta(x). \quad (2)$$

The degenerate problem A_0 $(\varepsilon = 0)$ consists in solving the equation

$$L_0 w \equiv -\varphi(x,t,w)\, \partial w/\partial t + \psi(x,t,w) = 0 \quad (3)$$

under the condition

$$w\big|_{t=0} = \alpha(x). \quad (4)$$

We assume that $\varphi(x,t,u) \geqslant a > 0$ for all points $(x,t,u) \in G \times \{-\infty < u < +\infty\}$. With the aid of the first iterative process we construct a sequence $w_0, w_1, \dots, w_n$ so that $L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1})$, where

$$\bar{w}_n = \sum_{i=0}^{n} \varepsilon^i w_i.$$

Expanding $\varphi(x,t,\bar{w}_n)$, $\psi(x,t,\bar{w}_n)$ at the point (x,t,w_0) in powers of ε and equating to zero the terms with like powers, we obtain

$$-\varphi_0(x,t,w_0)\,\partial w_0/\partial t + \psi_0(x,t,w_0) = 0, \qquad w_0\big|_{t=0} = \alpha(x); \tag{5}$$

$$\varphi_0(x,t,w_0)\,\partial w_k/\partial t + (\varphi_1\,\partial w_0/\partial t - \psi_1)\,w_k + \partial^2 w_{k-1}/\partial t^2 - \partial^2 w_{k-1}/\partial x^2 + U_k(x,t,w) = 0 \quad (k = 1,2,\dots,n), \tag{6}$$

where $U_k(x,t,w)$ depends only on w_i, $\partial w_i/\partial t$ $(i < k)$; $\varphi_0, \psi_0, \varphi_1, \psi_1$ are the coefficients of the expansions of the functions φ, ψ. The initial conditions for w_k will be determined below.

To find the asymptotics of the boundary layer

$$\bar{v}_n = \sum_{i=0}^{n+1} \varepsilon^i v_i$$

we introduce a new variable $y = t/\varepsilon$. We construct functions v_i $(i = 0,1,\dots,n+1)$ so that for the function $\bar{v}_n$ we have

$$L_\varepsilon(\bar{w}_n + \bar{v}_n) - L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1}). \tag{7}$$

Then, in view of $L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1})$, we obtain that

$$L_\varepsilon(\bar{w}_n + \bar{v}_n) = O(\varepsilon^{n+1}). \tag{8}$$

We rewrite equation (1) in the new variable

$$\varepsilon L_\varepsilon u = \varepsilon^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{\varepsilon^2} \frac{\partial^2 u}{\partial y^2} \right) - \varphi(x,\varepsilon y,u) \frac{\partial u}{\partial y} + \varepsilon \psi(x,\varepsilon y,u).$$

We expand the function $\bar{w}_n$ found above in a series in powers of $t = \varepsilon y$. Substituting its expression in (7) and expanding the functions $\varphi(x,\varepsilon y,\bar{w}_n)$, $\psi(x,\varepsilon y,\bar{w}_n)$ in powers of ε, we obtain successively

$$\partial v_0^2/\partial y^2 + \varphi_{00}(x,u_0+v_0)\,\partial v_0/\partial y = 0; \tag{9}$$

$$\partial^2 v_k/\partial y^2+\partial\left[\varphi_{00}(x,u_0+v_0)(u_k+v_k)-\varphi_{00}(x,u_0)u_k\right]/\partial y-(\bar{E}_k-E_k)=0,\quad (k=1,2,\dots,n+1), \tag{10}$$

where $\bar{E}_k - E_k$ depends only on $u_i,\ v_i,\ \partial u_i/\partial y,\ \partial v_i/\partial y,\ \partial^2 v_{k-1}/\partial x^2\ (i<k)$; $u_k,\ \varphi_{00}$ are the coefficients of the expansions of $\bar{w}_n,\ \varphi$.

Define v_k $(k=0,1,2,\dots,n+1)$ as solutions of equations (9), (10) under the following conditions:

$$\begin{gathered} v_0\big|_{y=0}=0, \qquad v_k\big|_{y=0}=-w_k\big|_{t=0}\quad (k=1,2,\dots,n); \qquad v_{n+1}\big|_{y=0}=0;\\ \partial v_0/\partial y\big|_{y=0}=\varepsilon\beta(x); \qquad \partial v_k/\partial y\big|_{y=0}=-\partial w_{k-1}/\partial t\big|_{t=0}\quad (k=1,2,\dots,n);\\ \partial v_{n+1}/\partial y\big|_{y=0}=-\partial w_n/\partial t\big|_{t=0}. \end{gathered} \tag{11}$$

For $\varphi \geq a > 0$ one can compute directly that

$$v_0=\varepsilon\beta(x)(1-e^{-cat/\varepsilon})/ca,$$

where $c=c(x)\geq 1$. By induction one can prove that all the functions v_k $(k=1,2,\dots,n+1)$ are functions of boundary-layer type.

Integrating equation (10) from $1/\varepsilon$ to y and using the fact that

$$\int_{1/\varepsilon}^{y}(\bar{E}_k-E_k)\,dy=B_k$$

is a function of boundary-layer type, we obtain the initial conditions for w_k:

$$\varphi_{00}(x,u_0)w_k\big|_{t=0}=-B_k\big|_{t=0}-\partial w_{k-1}/\partial t\big|_{t=0}\quad (k=1,2,\dots,n).$$

Theorem 1. *Let Q be any triangle in the domain G, bounded by characteristics of equation (1) and by a segment of the axis Ox. If: 1) in $G\times\{-\infty<u<+\infty\}$, φ,ψ have continuous derivatives of the form $\partial^m/\partial x^{i_1}\partial t^{i_2}\partial u^{i_3}$ $(m=1,2,\dots,2n+2)$; 2) $\varphi(x,t,u)\geq a>0$ for all $(x,t,u)\in G\times\{-\infty<u<+\infty\}$; 3) $\alpha(x)$ is continuously differentiable $2n$ $(n=2,3,\dots)$ times (for $n=0,1$ it has continuous derivatives up to the third order); 4) $\beta(x)$ has continuous derivatives up to order $(n-1)$ and $(n-2)$, respectively, for $n=2k+1,\ n=2k+2$ $(k=2,3,\dots)$ (for $n=0,1,2,3,4$, $\beta(x)$ is three times continuously differentiable); 5) $\varphi''_u,\psi''_t,\psi''_u$ are bounded in $Q\times\{-\infty<u<+\infty\}$, then: a) in Q there exists a unique solution $u_\varepsilon(x,t)$ of problem (1), (2), continuous and having the continuous derivatives entering equation (1); b) the first and second iterative processes converge; c) the solution $u_\varepsilon(x,t)$ admits the asymptotic representation*

$$u_\varepsilon=w_0+\varepsilon w_1+\dots+\varepsilon^n w_n+v_0+\varepsilon v_1+\dots+\varepsilon^{n+1}v_{n+1}+R_n,$$

where the w_i are obtained by means of the first iterative process; the v_i are functions of boundary-layer type near $t=0$, constructed by solving equations (9), (10) under conditions (11); the estimate $\|R_n\|_{L_2(Q)}=O(\varepsilon^{n+1})$ holds everywhere in Q.

2. Mixed problem

Problem A_ε. In a certain rectangle

$$R:\ \{0 \le x \le l,\ 0 \le t \le T\}$$

consider the following problem:

$$L_\varepsilon u_\varepsilon \equiv \varepsilon\,(\partial^2 u_\varepsilon/\partial x^2 - \partial^2 u_\varepsilon/\partial t^2) - \varphi(x,t,u_\varepsilon)\,\partial u_\varepsilon/\partial t + \psi(x,t,u_\varepsilon) = 0; \quad (12)$$

$$u_\varepsilon\big|_{t=0} = \alpha(x), \qquad \partial u_\varepsilon/\partial t\big|_{t=0} = \beta(x); \qquad u_\varepsilon(0,t) = u_\varepsilon(l,t) = 0;$$

$$\varepsilon > 0$$

is a small parameter.

The **degenerate problem** A_0 ($\varepsilon = 0$) consists in solving the problem

$$L_0 w \equiv -\varphi(x,t,w)\,\partial w/\partial t + \psi(x,t,w) = 0; \qquad w\big|_{t=0} = \alpha(x). \quad (13)$$

In the present case the first iterative process is unchanged, with the only difference from the Cauchy problem case that the initial conditions are defined as follows:

$$w_0\big|_{t=0} = \alpha(x), \qquad w_i\big|_{t=0} = 0 \quad (i = 1, 2, \dots, n).$$

We shall construct functions of boundary-layer type first near the sides $x = 0,\ x = l$, then near $t = 0$. In this case a parabolic boundary layer appears near $x = 0$ and $x = l$, i.e. the boundary-layer functions are constructed by solving parabolic equations. We shall restrict ourselves to the construction near $x = 0,\ 0 \le t \le T$, because the boundary-layer function near $x = l,\ 0 \le t \le T$ can be obtained by a similar method.

Introduce a new variable $y_1 = x/\sqrt{\varepsilon}$. We split the operator for constructing the boundary-layer functions $v_0^{(0)}, v_{1/2}^{(0)}, \dots, v_{n+1/2}^{(0)}, v_{n+1}^{(0)}$ in the same way as we did for the Cauchy problem. Hence we have

$$\partial^2 v_0^{(0)}/\partial y_1^2 - \varphi_{00}(t, \bar u_0 + v_0^{(0)})\,\partial v_0^{(0)}/\partial t + [\varphi_{00}(t, \bar u_0)$$

$$-\varphi_{00}(t, \bar u_0 + v_0^{(0)})]\partial^2 \bar u_0/\partial t + \psi_{00}(t, \bar u_0 + v_0^{(0)}) - \psi_{00}(t, \bar u_0) = 0; \quad (14)$$

$$v_0^{(0)}\big|_{t=0} = 0, \qquad v_0^{(0)}\big|_{y_1=0} = -\bar u_0(0,t) = -w_0(0,t); \quad (15)$$

$$\partial^2 v_{k/2}^{(0)}/\partial y_1^2 - \varphi_{00}(t, \bar{u}_0 + v_0^{(0)})\,\partial v_{k/2}^{(0)}/\partial t - [\varphi_{01}(t, \bar{u}_0 + v_0^{(0)})\,\partial(\bar{u}_0 + v_0^{(0)})/\partial t +$$

$$+\psi_{01}(t, \bar{u}_0 + v_0^{(0)})]v_{k/2}^{(0)} - F_{k/2} + \bar{E}_{k/2} - E_{k/2} = 0 \quad (k = 1, 2, \dots, 2n+2); \quad (16)$$

$$v_{i/2}^{(0)}\big|_{t=0} = 0 \quad (i = 1, 2, \dots, 2n+2); \qquad v_{i/2}^{(0)}\big|_{y_1=0} = 0$$

$$(i = 2k+1;\ k = 0, 1, \dots, n); \quad (17)$$

$$(\varepsilon v_k^{(0)} + \varepsilon \bar{u}_k + v_{k-1})\big|_{y_1=0} = 0 \quad (k = 1, 2, \dots, n+1),$$

where

$$F_{k/2} = [\varphi_{00}(t, \bar{u}_0 + v_0^{(0)}) - \varphi_{00}(t, \bar{u}_0)]\partial\bar{u}_{k/2}/\partial t +$$

$$+[\varphi_{01}(t, \bar{u}_0 + v_0^{(0)})\partial\bar{u}_0/\partial t + \varphi_{01}(t, \bar{u}_0 + v_0^{(0)})\partial v_0^{(0)}/\partial t$$

$$-\varphi_{01}(t, \bar{u}_0)\partial\bar{u}_0/\partial t - \psi_{01}(t, \bar{u}_0 + v_0^{(0)}) + \psi_{01}(t, \bar{u}_0)]\bar{u}_{k/2};$$

$\varphi_{00}, \varphi_{01}, \psi_{00}, \psi_{01}, \bar{u}_0, \bar{u}_{k/2}$ are the terms of the expansions of the functions $\varphi, \psi, \bar{w}_n$ in powers of $\sqrt{\varepsilon}$; $\bar{E}_{k/2} - E_{k/2}$ depends only on $\bar{u}_{i/2}, \partial\bar{u}_{i/2}/\partial t, v_{i/2}^{(0)}, \partial v_{i/2}^{(0)}/\partial t$ $(i < k)$ and $\partial^2 v_{k/2-1}^{(0)}/\partial t^2$, and we obtain

$$L_\varepsilon(\bar{w}_n + \bar{v}_n^{(0)}) - L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1}).$$

From $L_\varepsilon \bar{w}_n = O(\varepsilon^{n+1})$ it follows that

$$L_\varepsilon(\bar{w}_n + \bar{v}_n^{(0)}) = O(\varepsilon^{n+1}).$$

It can be proved that $v_0^{(0)}, v_{1/2}^{(0)}, \dots, v_{n+1/2}^{(0)}, v_{n+1}^{(0)}$ have the character of a boundary layer.

Analogously we can construct the boundary-layer function

$$\bar{v}_n^{(l)} = v_0^{(l)} + \sum_{i=1}^{2n+2} (\sqrt{\varepsilon})^i v_{i/2}^{(l)}$$

near $x = l,\ 0 \le t \le T$. Suppose that

$$\tilde{v}_n^{(0)} = \psi_1(x/\delta)\,\bar{v}_n^{(0)}, \qquad \tilde{v}_n^{(l)} = \psi_2((l-x)/\delta)\,\bar{v}_n^{(l)},$$

where ψ_1, ψ_2 are smoothing functions. Then everywhere in R one has

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)}) = O(\varepsilon^{n+1}), \qquad L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(l)}) = O(\varepsilon^{n+1}).$$

Now we pass to the construction of the sequence of boundary-layer functions near $t = 0,\ 0 \le x \le l$. They are constructed in the same way as for the Cauchy problem, and we note only that in the present case, in a neighborhood of $t = 0$, we expand not only the functions $w_0, w_1, \dots, w_n$, but also the boundary-layer functions $\tilde{v}_n^{(0)}, \tilde{v}_n^{(l)}$. Suppose that the expansions have the form

$$\tilde{v}_n^{(0)} = U_0^{(0)} + \varepsilon U_1^{(0)} + \dots + \varepsilon^{n+1} U_{n+1}^{(0)} + \varepsilon^{n+2} U_{n+2}^{(0)},$$

$$\tilde{v}_n^{(l)} = U_0^{(l)} + \varepsilon U_1^{(l)} + \dots + \varepsilon^{n+1} U_{n+1}^{(l)} + \varepsilon^{n+2} U_{n+2}^{(l)}.$$

By virtue of the conditions (17) and the conditions of the theorem (conditions 1 and 7), the terms $U_i^{(0)}, U_i^{(l)}$ $(i = 0, 1, 2, \dots, n+1)$ are equal to zero, while $U_{n+2}^{(0)}, U_{n+2}^{(l)}$ are bounded. Therefore, when splitting the operator in a neighborhood of $t = 0$, one may regard either $\bar{v}_n^{(0)}$ or $\tilde{v}_n^{(l)}$ as included in $O(\varepsilon^{n+2})$. Hence we have

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)} + \bar{v}_n) - L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)}) = O(\varepsilon^{n+1}),$$

where $\bar{v}_n = \psi_3(t/\delta)\bar{v}_n$; $\bar{v}_n$ is the boundary-layer function constructed near $t = 0,\ 0 \le x \le l$; ψ_3 is a smoothing function. It is already known that

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)}) = O(\varepsilon^{n+1}).$$

From this it follows that

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)} + \bar{v}_n) = O(\varepsilon^{n+1}).$$

Analogously we have

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(l)} + \bar{v}_n) = O(\varepsilon^{n+1}).$$

By virtue of the property of the smoothing function, everywhere in R one has

$$L_\varepsilon(\bar{w}_n + \tilde{v}_n^{(0)} + \tilde{v}_n^{(l)} + \bar{v}_n) = O(\varepsilon^{n+1}).$$

Theorem 2. *If: 1) in $R \times \{-\infty < u < +\infty\}$ the functions φ, ψ are continuous together with their derivatives of the form $\partial^m/\partial x^{i_1} \partial t^{i_2} \partial u^{i_3}$ $(m = 1, 2, \ldots, 3n+4)$; 2) on $[0, l]$ $a(x)$ is continuously differentiable $2n+3$ times; 3) $\beta(x)$ is continuously differentiable $n-1$, $n-2$ times, respectively, for $n = 2k+1$,*

$n = 2k+6$ $(k = 2, 3, \ldots)$ (for $n = 0, 1, 2, 3, 4$ it is continuous together with its derivatives of the second and third orders on $[0, l]$; 4) $\alpha(0) = \alpha(l) = \beta(0) = \beta(l) = 0$, $\alpha'(0) = \alpha'(l) = 0$, $\alpha''(0) = -\psi(0,0,0)$, $\alpha''(l) = -\psi(l,0,0)$; 5) the functions $\varphi(x,t,u)$, $\psi(x,t,u)$ with respect to the arguments t, u have derivatives of the first and second orders, bounded in $R_t \times \{-\infty < u < +\infty\}$, where $0 \le t^* \le T$; 6) in $R \times \{-\infty < u < +\infty\}$ $\varphi(x,t,u) \ge a > 0$; 7) $\partial^m \psi/dx^{i_1} dt^{i_2} = 0$ at the points $(0,0,0)$, $(l,0,0)$ $(m = 0, 1, 2, \ldots, n+2;\ i_1 = 0, 1, 2, 4, 6, \ldots)$; 8) $\varphi'_u w'_t(0,t) > \psi'_u$ for $x = 0$ in $[0,T] \times \{-\infty < u < +\infty\}$, where w is the solution of the degenerate problem, then: a) there exists[4] a unique solution of problem (1), (2), having continuous derivatives entering the equation; b) problem (14), (15) has[5] a unique and bounded solution u; c) the first and second iteration processes converge; d) the solution $u_\varepsilon(x,t)$ admits the asymptotic representation

$$u_\varepsilon = \bar{w}_n + \tilde{v}_n^{(0)} + \tilde{v}_n^{(l)} + \tilde{v}_n + R_n, \quad \text{where } \bar{w}_n = w_0 + \sum_{i=1}^{n} \varepsilon^i w_i;$$

w_i are obtained by means of the first iteration process;

$$\tilde{v}_n^{(0)} = \psi_1(x/\delta)\bar{v}_n^{(0)} = \psi_1(x/\delta)\left(v_0^{(0)} + \sum_{i=1}^{2n+2} (\sqrt{\varepsilon})^i v_{i/2}^{(0)}\right);$$

$v_{i/2}^{(0)}$ are functions of the parabolic boundary layer near $x = 0$, $0 \le t \le T$;

$$\tilde{v}_n^{(l)} = \psi_2((l-x)/\delta)\left(v_0^{(l)} + \sum_{i=1}^{2n+2} (\sqrt{\varepsilon})^i v_{i/2}^{(l)}\right);$$

$v_{i/2}^{(l)}$ are near $x = l$, $0 \le t \le T$;

$$\tilde{v}_n = \psi_3(t/\delta)\bar{v}_n = \psi_3(t/\delta)\left(v_0 + \sum_{i=1}^{n+1} \varepsilon^i v_i\right);$$

v_i are functions of the boundary layer near $t = 0$, $0 \le x \le l$; the estimate

$$\|R_n\|_{L_2(R)} = O(\varepsilon^{n+1})$$

holds everywhere in R.

3. The case when the unperturbed equation is of the same order as the perturbed one.

A. An elliptic equation degenerates into a parabolic one. In the rectangle $R: (0 \leq x \leq l,\ 0 \leq t \leq T)$ the following problem is considered:

$$L_\varepsilon u_\varepsilon \equiv \varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{\partial^2 u_\varepsilon}{\partial x^2} - \varphi(x, t, u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} + \psi(x, t, u_\varepsilon) = 0;$$

$$u_\varepsilon|_\Gamma = 0,$$

where Γ is the boundary of R.

B. A hyperbolic equation degenerates into a parabolic one. In the domain

$$G: \{-\infty < x < +\infty,\ t \geq 0\}$$

we consider the Cauchy problem

$$L_\varepsilon u_\varepsilon \equiv -\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{\partial^2 u_\varepsilon}{\partial x^2} - \varphi(x, t, u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} + \psi(x, t, u_\varepsilon) = 0;$$

$$u_\varepsilon|_{t=0} = \alpha(x), \qquad \left.\frac{\partial u_\varepsilon}{\partial t}\right|_{t=0} = \beta(x).$$

The asymptotics of the solutions of these problems is constructed in the same way as for the Cauchy problem in § 1, except that in the first iteration process $w_0, w_1, \ldots, w_n$ are obtained as solutions of parabolic equations.

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Note: Figure translations are in progress. See original paper for figures.