Reports of the Academy of Sciences of the USSR

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1961

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1961. Volume 140, No. 5

MATHEMATICS

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ON THE BENDING OF SURFACES OF POSITIVE CURVATURE UNDER CERTAIN BOUNDARY CONDITIONS

(Presented by Academician P. S. Aleksandrov, 18 V 1961)

The question of the existence of surfaces isometric to a given one reduces, as is well known, to the problem of the existence of solutions of the system of Gauss and Peterson–Codazzi equations, where, for a prescribed metric, the role of the unknown functions is played by the coefficients of the second quadratic form. On a surface with boundary, by prescribing some characteristic of the contour (curvature, geodesic torsion), we subject the sought functions to an additional boundary condition. If the resulting boundary-value problem admits a solution with a known arbitrariness, then one may speak of a bending of the surface isometric to the original one. The latter, in this case, need not belong to a continuous family of isometric surfaces.

In the present note we consider certain conditions for the existence of solutions of the indicated boundary-value problem and the resulting criteria for the non-bendability of pieces of surfaces of positive curvature.

Let a piece of a surface S of positive Gaussian curvature, belonging to the class C^3_α ($\alpha < 1$), satisfy the following conditions:

1) An isometrically conjugate parametrization of S maps it homeomorphically onto a domain D (generally speaking, $(m+1)$-connected) with boundary
$L = L_1 + ... + L_{m+1}$
of the parametric plane.

2) The domain D, without loss of generality, will henceforth be assumed canonical.

Let $\Delta L, \Delta M, \Delta N$ be the increments of the coefficients of the second form, and $\Delta k_n, \Delta \tau_g$, respectively, those of the normal curvature and the geodesic torsion of the boundary under passage to an isometric surface. Introduce new unknown functions U, V, Π, related to $\Delta L, \Delta M, \Delta N$ by the formulas

$$\Delta L = \Pi + V, \qquad \Delta M = U, \qquad \Delta N = \Pi - V. \tag{1}$$

The Codazzi equations will then be equivalent to the following system:

$$U_x - V_y = \alpha U + \beta V + L_0 \left(\frac{\Pi}{L_0}\right)_y,$$

$$U_y + V_x = \gamma U + \delta V + L_0 \left(\frac{\Pi}{L_0}\right)_x; \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are known functions, invariant under bending, and L_0 is the coefficient of the second form of the original surface.

As a consequence of the Gauss equation, the functions U, V, Π must satisfy the algebraic relation

$$U^2 + V^2 - \Pi^2 = 2L_0\Pi. \tag{3}$$

Introduce on the contour two functions $\lambda(r,\varphi)$ and $\mu(r,\varphi)$, belonging to the class $C^1_\alpha(L)$, with the aid of which, using the contour relation of the sought functions

with the quantities Δk_n and $\Delta \tau_g$, we obtain the following boundary condition:

$$U\left(-\mu\frac{g}{e} + \lambda\sin 2\varphi + \mu\cos 2\varphi\right) + V\left(\mu\frac{f}{e} + \lambda\cos 2\varphi - \mu\sin 2\varphi\right)$$
$$= \Pi\left(\mu\frac{f\cos 2\varphi - g\sin 2\varphi}{e} + \lambda\right) - p\left(\frac{\sqrt{a}}{e}\mu\Delta\tau_g + \lambda\Delta k_n\right). \tag{4}$$

Here

$$a = EG - F^2, \qquad p(r,\varphi) = E\sin^2\varphi - 2F\sin\varphi\cos\varphi + G\cos^2\varphi,$$

$$e = \frac{1}{2}(E+G), \qquad g = \frac{1}{2}(E-G), \qquad f = F.$$

Theorem 1. Let D be a canonical domain; L its boundary; let $\lambda(r,\varphi)$, $\mu(r,\varphi)$, $\sigma(r,\varphi)$ be functions prescribed on L, belonging to $C^1_\alpha(L)$ and satisfying the conditions:

$$\varkappa = \operatorname{Ind}(\mu + i\lambda) > -(m-1), \qquad C^1_\alpha(\sigma) < \varepsilon$$

(ε is a sufficiently small positive constant, the norm being defined as in ([1])).

Then the system of equations (2), (3), (4) has precisely a $(2\varkappa{-}3{+}3m)$-parameter family of solutions belonging to $\overline{C}^1_\alpha(D+L)$ such that

$$\frac{\sqrt{a}}{e}\mu(r,\varphi)\Delta\tau_g(r,\varphi)+\lambda(r,\varphi)\Delta k_n(r,\varphi)=\sigma(r,\varphi). \tag{5}$$

Proof. By the methods of ([1]) one can obtain the general solution

$$w(x,y,\Pi)=U(x,y,\Pi)+iV(x,y,\Pi)$$

of the system (2), (4), in which the function $\Pi(x,y)\in C^1_\alpha(D+L)$ is regarded as given. The latter must be such that the integro-differential equation, obtained by substituting $w(x,y)$ into (3), with respect to the now unknown function $\Pi(x,y)$,

$$|w|^2-\Pi^2=2L_0\Pi \tag{6}$$

is solvable. The solvability of (6) is proved by the method of successive approximations under the assumption $\Pi^0=0$. The solution of (6) is unique.

Corollary. For a simply connected piece of a surface, under the conditions of Theorem 1, there exists a $(2\varkappa-3)$-parameter family of isometric surfaces satisfying the boundary condition (5).

Theorem 2. Let $\lambda(r,\varphi)$, $\mu(r,\varphi)\subset C^1_\alpha(L)$ satisfy the condition

$$\mathrm{Ind}(\mu,\lambda)<-2(m-1).$$

For the piece S to be non-bendable, it is sufficient that the increment of some linear combination of k_n and τ_g, prescribed on the contour, be equal to zero:

$$\frac{\sqrt{a}}{e}\mu(r,\varphi)\Delta\tau_g(r,\varphi)+\lambda(r,\varphi)\Delta k_n(r,\varphi)=0.$$

The assertion of the theorem follows from the fact that, for

$$\varkappa<-2(m-1)$$

and $\sigma(r,\varphi)=0$, equation (6) has the unique solution $\Pi\equiv 0$, and the system (2), (4) admits no solutions other than the trivial ones.

In particular, for $m=0$ one obtains sufficient conditions for non-bendability:
a) $\Delta\tau_g=0$ (K. M. Belov ([3])),
b) $\Delta k_n=\Delta\tau_g$,
c) $\Delta k_n=0$. The last case we formulate in the form of the following theorem:

Theorem 3. If between two pieces of surfaces of positive curvature of class C^3_α an isometric correspondence is established, under which the curvature of corresponding boundary points is the same, then such surfaces are congruent or symmetric.

The condition $\Delta k = 0$ can be realized by means of a construction consisting of two surfaces of positive curvature glued along a common edge in such a way that the gluing angle, different from zero at every point, does not change under continuous isometric deformations. Such a construction is uniquely determined. Moreover, the theorem on ova-

to people with a hemmed edge for $m = 0$ ([1]) carries over to continuous bendings.

By the method presented above, relying on Nitsche' s theorem ([4]), one can prove the following theorem:

Theorem 4. Let $\lambda(\varphi), \mu(\varphi) \subset C^1_\alpha(L)$ be prescribed on the contour, satisfying the condition $\chi = \mathrm{Ind}(\mu + i\lambda) \geq 2$, and

$$h_k(\varphi) = 1, \cos\varphi, \sin\varphi, \cos 2\varphi, \sin 2\varphi, \dots .$$

For a simply connected piece of a surface S of positive curvature of class C^3_α to be rigid, it is necessary and sufficient that the following conditions be satisfied on its contour:

$$\Delta\left[\frac{\sqrt{a}}{e}\mu(\varphi)\tau_g(\varphi) + \lambda(\varphi)k_n(\varphi)\right] = 0,$$

$$\int_0^{2\pi} \frac{\chi(\varphi)}{\sqrt{\lambda^2+\mu^2}}\left[\mu(\varphi)\Delta k_n(\varphi) - \lambda(\varphi)\frac{\sqrt{a}}{e}\Delta\tau_g(\varphi)\right] h_k(\varphi)\, d\varphi = 0$$

$$(k = 1, \dots, 2\chi - 3),$$

where

$$\chi(\varphi) = \frac{E\sin^2\varphi - 2F\sin\varphi\cos\varphi + G\cos^2\varphi}{\left[-\sin(\theta-2\varphi) + \frac{g}{e}\sin\theta\right]^2 + \left[\cos(\theta-2\varphi) - \frac{f}{e}\sin\theta\right]^2},$$

$$\sin\theta = \frac{\lambda}{\sqrt{\lambda^2+\mu^2}}, \qquad \cos\theta = \frac{\mu}{\sqrt{\lambda^2+\mu^2}}.$$

Finally, let us consider the bending of a belt $\widetilde{S}$ of a surface of revolution of the second order of positive curvature, under the condition that the geodesic torsion is prescribed on the edge ($\lambda = 0$, $\mu = 1$). The case of bending a spherical belt into a surface with a prescribed constant normal curvature of the edge ($\lambda = 1,\ \mu = 0$) was investigated by Nitsche ([5]).

Theorem 5. Let the belt $\widetilde{S}$, mapped onto the annulus D with boundary $L = L_1 + L_2$, satisfy the following requirements:

$$\Delta\tau_g(\varphi) \in C'_\alpha(L), \qquad \Delta k_{n_2}(\varphi) \in C'_\alpha(L_2),$$

$$\left|\int_{-\pi}^{\pi} \Delta k_{n_1}(\varphi)\, d\varphi\right| < \varepsilon, \qquad C'_\alpha(\Delta\tau_g) < \varepsilon,$$

where ε is a sufficiently small positive number. For there to exist no more than a one-parameter family of surfaces, isometric to $\widetilde{S}$, for which the geodesic torsion of the edge assumes a prescribed value $\tau_g(r, \varphi)$, it is necessary and sufficient that the contour condition be fulfilled

$$\sqrt{f_1(1) f_2(1)} \int_{-\pi}^{\pi} \sqrt[4]{K_1}\, \Delta\tau_{g_1}(\varphi)\, d\varphi = q^4 \sqrt[4]{f_1(q) f_2(q)} \int_{-\pi}^{\pi} \sqrt[4]{K_2}\, \Delta\tau_{g_2}(\varphi)\, d\varphi,$$

where $f_i(r)$ are known functions of $r = \sqrt{x^2 + y^2}$, invariant under bending, $r_1 = 1,\ r_2 = q < 1$, and K_i is the Gaussian curvature of the surface along the corresponding contour.

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Received
31 III 1961

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Note: Figure translations are in progress. See original paper for figures.