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Abstract

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ON SOME PROPERTIES OF LEVEL LINES UNDER UNIVALENT CONFORMAL MAPPINGS

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In a preceding joint paper ([1]) the authors proved that, under a univalent conformal mapping of the disk $|z| < 1$, realized by a function $w = f(z)$ regular in this disk, the number of inflection points of the level lines L_r (the image of the circle $|z| = r$) and the number of points at which its starlikeness is violated (points of the level line at which the direction of rotation of the radius vector changes when z traverses the circle $|z| = r$ in a fixed direction) may vary nonmonotonically as z approaches the boundary of the unit disk; i.e., if $r_1 < r_2$, it may turn out that the level line L_{r_1} has more inflection points or more points at which starlikeness is violated than the level line L_{r_2}. The examples given in the cited paper consisted of functions mapping the disk $|z| < 1$ onto bounded domains.

The purpose of the present paper is to show that, when the modulus of a function univalent in the disk $|z| < 1$ grows sufficiently rapidly, a certain regularity must already be observed in the behavior of its level lines as $r \to 1$. Namely, the following theorems hold.

Theorem 1. For the class S of functions

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n,$$

regular and univalent in the disk $|z| < 1$, there exists an absolute constant α_s, $0.1005 < \alpha_s < 0.134$, such that every arc of the level line L_r of any function $f(z) \in S$, lying in the annulus

$$\alpha_s \frac{r}{(1-r)^2} < |f(z)| < \frac{r}{(1-r)^2}, \qquad |z| = r < 1,$$

is starlike; however, there exist functions $f(z) \in S$ for which some arc of a level line lying in the wider annulus

$$(\alpha_s - \varepsilon) \frac{r}{(1-r)^2} < |f(z)| < \frac{r}{(1-r)^2}, \qquad \varepsilon > 0,$$

is no longer starlike, if r is sufficiently close to 1*.

Theorem 2. For the class S of functions

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n,$$

regular and univalent in the disk $|z| < 1$, there exists an absolute constant α_k, $0.333\ldots < \alpha_k < 0.511$, such that every arc of the level line L_r of any function $f(z) \in S$, lying in the annulus

$$\alpha_k \frac{r}{(1-r)^2} < |f(z)| < \frac{r}{(1-r)^2}, \qquad |z| = r < 1,$$

* The quantity $r/(1-r)^2$ is the exact upper bound for the modulus $|f(z)|$, $|z| \le r < 1$, in the class S.

is convex, but there exist functions $f(z) \in S$ for which some arc of a level line lying in the wider annulus

$$(a_k - \varepsilon) \frac{r}{(1-r)^2} < |f(z)| < \frac{r}{(1-r)^2}, \qquad \varepsilon > 0,$$

will no longer be convex, if r is sufficiently close to 1.

Analogous theorems are proved for the class Σ of functions

$$F(\zeta) = \frac{1}{f(1/\zeta)} = \zeta + \sum_{n=0}^{\infty} \frac{C_{-n}}{\zeta^n}, \qquad f(z) \in S, \qquad \zeta = \frac{1}{z},$$

univalent and regular in the domain $|\zeta| > 1$, except for the simple pole $\zeta = \infty$.

Theorem 3. For the class Σ of functions $F(\zeta) = \dfrac{1}{f(1/\zeta)}$, $f(z) \in S$, $\zeta = 1/z$, there exists an absolute constant $A_s = 1/a_s$, $7.667 < A_s < 10$, such that every arc of the level line L_ρ ($\rho = |\zeta|$) of any function $F(\zeta) \in \Sigma$, lying in the annulus

$$\frac{(\rho-1)^2}{\rho} < |F(\zeta)| < A_s \frac{(\rho-1)^2}{\rho}, \qquad |\zeta| = \rho > 1,$$

is starlike; but there exist functions $F(\zeta) \in \Sigma$ for which some arc of a level line lying in the wider annulus

$$\frac{(\rho-1)^2}{\rho} < |F(\zeta)| < (A_s + \varepsilon) \frac{(\rho-1)^2}{\rho}, \qquad \varepsilon > 0,$$

will no longer be starlike, if ρ is sufficiently close to 1*.

Theorem 4. For the class Σ of functions $F(\zeta) = \dfrac{1}{f(1/\zeta)}$, $f(z) \in S$, $\zeta = \dfrac{1}{z}$, there exists an absolute constant A_k, $1.75 < A_k < 10$, such that every arc of the level line L_ρ of any function $F(\zeta) \in \Sigma$, lying in the annulus

$$\frac{(\rho-1)^2}{\rho} < |F(\zeta)| < A_k \frac{(\rho-1)^2}{\rho}, \qquad |\zeta| = \rho > 1,$$

is convex, but there exist functions $F(\zeta) \in \Sigma$ for which some arc of a level line lying in the wider annulus

$$\frac{(\rho-1)^2}{\rho} < |F(\zeta)| < (A_k + \varepsilon)\frac{(\rho-1)^2}{\rho}, \qquad \varepsilon > 0,$$

will no longer be convex, if ρ is sufficiently close to 1.

Moscow Institute of Steel
named after I. V. Stalin

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1. I. E. Bazilevich, G. V. Koritskii, *Mat. sborn.*, **32** (74), 1, 209 (1953).

* The quantity $(\rho-1)^2/\rho$ is the exact lower bound of the modulus $|F(\zeta)|$, $|\zeta| \geq \rho - 1$, in the class Σ.

Note: Figure translations are in progress. See original paper for figures.