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Abstract

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MATHEMATICS

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A CRITERION FOR THE UNICELLULARITY OF VOLTERRA OPERATORS

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Let A be a Volterra* operator acting in a separable Hilbert space $\mathfrak{H}$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are invariant subspaces of it ($\mathfrak{H}_1 \subset \mathfrak{H}_2$, $\dim\{\mathfrak{H}_2 \ominus \mathfrak{H}_1\} > 1$), then, by the theorem of Aronszajn and Smith ([1]), there exists an invariant subspace $\mathfrak{H}_3$ such that $\mathfrak{H}_1 \subset \mathfrak{H}_3 \subset \mathfrak{H}_2$. The operator A is called **unicellular** ([2]) if, of any two invariant subspaces of it, one belongs to the other. In the present paper a necessary and sufficient condition is established for the unicellularity of an operator A, which is satisfied by its characteristic operator-function.

1. Generalizing a definition of M. S. Livšic ([3]), represent the imaginary part of the operator A in the form

$$\frac{A - A^*}{2i} = RJR^*,$$

where R is a completely continuous mapping of some Hilbert space $\mathfrak{H}_W$ into $\mathfrak{H}$, and J is an operator acting in $\mathfrak{H}_W$ and satisfying the conditions $J = J^*$, $J^2 = E$. The operator-function

$$W(\lambda) = E - 2iR_*(A - \lambda E)^{-1}RJ \tag{1}$$

is called the **characteristic** operator-function for the operator A. If $\mathfrak{H}_0$ is some subspace in $\mathfrak{H}$, then the operator-function

$$W_0(\lambda) = E - 2iR_0^*(A_{P_0} - \lambda E)^{-1}R_0J \quad (A_{P_0}f = P_0Af\ (f \in \mathfrak{H}_0),\ R_0 = P_0R), \tag{2}$$

where P_0 is the projection operator onto $\mathfrak{H}_0$, is called the **projection** of $W(\lambda)$ onto $\mathfrak{H}_0$ and is denoted by the symbol $\mathrm{Pr}_{\mathfrak{H}_0} W(\lambda)$.

From the general theory of characteristic functions ([4,5]) the following assertion follows:

Lemma 1. *If in the space $\mathfrak{H}$ there exist two distinct subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, invariant with respect to A, such that*

$$\mathrm{Pr}_{\mathfrak{H}_1} W(\lambda) = \mathrm{Pr}_{\mathfrak{H}_2} W(\lambda),$$

then the operator A commutes with some nonscalar unitary operator.

We shall also need:

Lemma 2. *If the operator A commutes with some nonscalar unitary operator U, then*

$$\mathfrak{H} = \mathfrak{H}^{(1)} \oplus \mathfrak{H}^{(2)} \quad (\mathfrak{H}^{(k)} \neq 0,\ k = 1, 2),$$

where $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ are invariant with respect to A.

Proof. Denote by $\mathfrak{H}_0$ the totality of all eigenvectors of the operator

$$\frac{A - A^*}{2i},$$

corresponding to some nonzero eigenvalue. Since $\mathfrak{H}_0$ is finite-dimensional and invariant with respect to U, there exists in $\mathfrak{H}_0$ an eigenvector f of the operator U. Putting $Uf = tf$, denote by $\mathfrak{H}^{(1)}$ the subspace consisting of all

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* The operator A is called **Volterra** if it is completely continuous and has no nonzero points of the spectrum.

vectors g for which $Ug = \tau g$. It is easy to see that $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)} = \mathfrak{H} \ominus \mathfrak{H}^{(1)}$ are invariant with respect to A.

2. The operator-function $W(\lambda)$ has the following properties:

I. The function $W(\lambda)$ expands into a norm-convergent series

$$W(\lambda) = E + \frac{1}{\lambda} W_1 + \frac{1}{\lambda^2} W_2 + \cdots \quad (\lambda \neq 0),$$

where W_k $(k = 1, 2, \ldots)$ are completely continuous operators.

II. $W^*(\lambda) J W(\lambda) - J \geq 0,\ \operatorname{Im} \lambda > 0; \quad W^*(\lambda) J W(\lambda) - J = 0,\ \operatorname{Im} \lambda = 0,\ \lambda \neq 0.$

We shall denote by (Ω_J) the totality of all operator-functions satisfying conditions I and II. It can be shown that each function of this class is characteristic for some Volterra operator.

Let $W_1(\lambda)$ and $W_2(\lambda)$ belong to the class (Ω_J). The notation $W_2(\lambda) < W_1(\lambda)$ will mean that there exists a function $W_3(\lambda) \in (\Omega_J)$ such that $W_1(\lambda) = W_2(\lambda) W_3(\lambda)$. In this case we shall say that $W_2(\lambda)$ is a divisor of the function $W_1(\lambda)$. If the subspace $\mathfrak{H}_0$ is invariant with respect to A, then

$$W(\lambda) = \operatorname{Pr}_{\mathfrak{H}_0} W(\lambda) \operatorname{Pr}_{\mathfrak{H} \ominus \mathfrak{H}_0} W(\lambda) \tag{6}$$

where the functions $\operatorname{Pr}_{\mathfrak{H}_0} W(\lambda)$ and $\operatorname{Pr}_{\mathfrak{H} \ominus \mathfrak{H}_0} W(\lambda)$ are characteristic, respectively, for the Volterra operators $P_0 A f$ $(f \in \mathfrak{H}_0)$ and $(E - P_0) A f$ $(f \in \mathfrak{H} \ominus \mathfrak{H}_0)$, and therefore belong to the class (Ω_J). Thus to every invariant subspace of the operator A there corresponds a divisor of its characteristic operator-function. It is easy to verify that the converse assertion is false.

An operator-function of the form

$$T(\lambda) = E + \frac{2i\sigma}{\lambda} PJ \quad (\sigma > 0,\ PJP = 0),$$

where P is the operator of projection onto a one-dimensional subspace in $\mathfrak{H}_W$, will be called elementary. Clearly, all elementary functions belong to the class (Ω_J). Two elementary functions

$$T_1 = E + \frac{2i\sigma_1}{\lambda} P_1 J, \qquad T_2 = E + \frac{2i\sigma_2}{\lambda} P_2 J$$

we shall agree to call similar if $P_1 = P_2$.

Let $W(\lambda) = W_1(\lambda) W_2(\lambda)$ $(W_k(\lambda) \in (\Omega_J))$. We shall say that $W_1(\lambda)$ is a proper divisor of the function $W(\lambda)$ if $W_1(\lambda)$ and $W_2(\lambda)$ cannot be represented in the form

$$W_1(\lambda) = W_{11}(\lambda) W_{22}(\lambda), \qquad W_2(\lambda) = W_{21}(\lambda) W_{22}(\lambda) \quad (W_{ij}(\lambda) \in (\Omega_J)),$$

where $W_{12}(\lambda)$ and $W_{21}(\lambda)$ are similar elementary functions.

Lemma 3. *If the set of vectors of the form A^{nRf} $(f \in \mathfrak{H}_W,\ n = 0, 1, 2, ...)$ is complete in $\mathfrak{H}$, then a divisor $W_1(\lambda)$ of the function $W(\lambda)$ coincides with the projection of this function onto some invariant subspace of the operator A if and only if it is proper.*

Lemma 4. *If $W_1(\lambda)$ is an improper divisor of the function $W(\lambda)$, then there exist proper divisors $W_2(\lambda)$ and $W_3(\lambda)$ such that $W_2(\lambda) < W_1(\lambda) < W_3(\lambda)$ and $W_3(\lambda) = W_2(\lambda) W_4(\lambda)$, where $W_4(\lambda)$ is an elementary function.*

3. A characteristic operator-function $W(\lambda)$ will be called ordered if, for any two of its divisors, one is a divisor of the other.

Theorem. *A simple* Volterra operator A if and only if

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* A Volterra operator A is called simple if in $\mathfrak{H}$ there is no subspace annihilated by the operators A and A^*. If A is a simple Volterra operator, and P_0 is the projector onto its invariant subspace $\mathfrak{H}_0$, then the set of vectors of the form A^{nRf} $(f \in \mathfrak{H}_W,\ n = 0, 1, 2, ...)$ is complete in $\mathfrak{H}$, and the set of vectors of the form $A_0^{nP} Rf$ $(f \in \mathfrak{H}_W,\ n = 0, 1, 2, ...)$ is complete in $\mathfrak{H}_0$.

case it is unicellular when its characteristic operator-function $W(\lambda)$ is ordered.

Proof. Let $W(\lambda)$ be an ordered function. If the operator A is not unicellular, then there exist subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, invariant with respect to A, such that neither of them belongs to the other. The projections $W_1(\lambda)$ and $W_1'(\lambda)$ of the function $W(\lambda)$ onto these subspaces are comparable with one another. Suppose, for example, $W_1(\lambda) < W_1'(\lambda)$. From Lemma 3 and the relations

$$W_1'(\lambda) = W_1(\lambda)W_2(\lambda), \quad W(\lambda) = W_1(\lambda)W_2(\lambda)W_3(\lambda) \quad (W_2(\lambda), W_3(\lambda) \in (\Omega_J)),$$

it follows that $W_1(\lambda)$ is a proper divisor of the function $W_1'(\lambda)$. Applying Lemma 3 again, we find an A-invariant subspace $\mathfrak{H}_2 \subset \mathfrak{H}_1$ such that the projection of $W(\lambda)$ onto $\mathfrak{H}_2$ will be equal to $W_1(\lambda)$. Since $\mathfrak{H}_2 \neq \mathfrak{H}_1$, by Lemmas 1 and 2 the space $\mathfrak{H}$ is representable in the form $\mathfrak{H} = \mathfrak{H}^{(1)} \oplus \mathfrak{H}^{(2)}$ $(\mathfrak{H}^{(k)} \neq 0,\ k = 1, 2)$, where $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ are invariant with respect to A.

Introduce the projection operator $P^{(k)}$ $(k = 1, 2)$ onto the subspace $\mathfrak{H}^{(k)}$, and consider the projections

$$W^{(k)}(\lambda) = E - 2iR^*P^{(k)}(A_{P^{(k)}} - \lambda E)^{-1}P^{(k)}RJ \quad (k = 1, 2).$$

One of these projections, for instance $W^{(1)}(\lambda)$, must be a divisor of the other,

$$W^{(2)}(\lambda) = W^{(1)}(\lambda)W^{(3)}(\lambda) \quad (W^{(3)}(\lambda) \in (\Omega_J)).$$

Expanding both sides of the last equality in a series in a neighborhood of the point at infinity and comparing the coefficients of $1/\lambda$, we obtain

$$P^*P^{(2)}R = R^*P^{(1)}R + H \quad (H \geq 0). \tag{3}$$

Neither of the subspaces $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ can be annihilated by the operator $\dfrac{A - A^*}{2i}$, since otherwise the operator A would not be simple. Consequently, in the subspace $\mathfrak{H}^{(1)}$ there is an eigenvector e $(\|e\| = 1)$ of the operator $\dfrac{A - A^*}{2i}$, corresponding to a nonzero eigenvalue ω. Thus

$$(RJHJR^*e, e) = \left(\frac{A - A^*}{2i}(P^{(2)} - P^{(1)})\frac{A - A^*}{2i}e, e\right) = -\omega^2,$$

which contradicts equality (3).

Now let A be a unicellular operator. Then the comparability of two proper divisors of the function $W(\lambda)$ is obvious. The comparability of divisors in the remaining cases follows easily from Lemma 4.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.