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Abstract

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ON ANALYTIC SOLUTIONS OF SOME NON-LINEAR INTEGRAL EQUATIONS

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In the space C of complex-valued functions continuous on the interval $[0,1]$, consider the equation

$$\varphi = \lambda\,[K_1\varphi + K_n\varphi^n + K_{n+1}\varphi^{n+1} + \cdots], \tag{1}$$

where λ is a complex parameter,

$$K_i\varphi^i = \int_0^1 K_i(x,s)\varphi^i(s)\,ds, \tag{2}$$

and the series

$$K_1(x,s)z + K_n(x,s)z^n + K_{n+1}(x,s)z^{n+1} + \cdots$$

converges for $0 \le x, s \le 1,\ |z| < R$ to a function $K(x,s,z)$ continuous jointly in the variables x, s, z and analytic in z.

Suppose that $K_1(x,s)$ has unity as an eigenvalue. As is known ([1]), this condition is necessary in order that equation (1) have nonzero solutions tending to zero as the parameter tends to unity. To obtain sufficient conditions for the existence of such solutions, both analytic methods (for example ([2–5])) and topological ones ([1]) have been used.

In the present note the Nekrasov–Nazarov process for constructing solutions of equation (1) is investigated. It turns out that, in the case when unity is a multiple eigenvalue of the kernel $K_1(x,s)$, the scheme of successive determination of the coefficients may have certain features not encountered in the case of a simple eigenvalue. Further, M. A. Krasnosel' skii' s assumptions (stated in a more general form) are proved concerning the indices of solutions constructed by the Nekrasov–Nazarov method, and concerning the index of any isolated solution of equation (1). In conclusion we show that, under certain conditions, all small solutions of (1) for values of λ sufficiently close to 1 are algebraic functions of λ.

1. By assumption, unity is an eigenvalue of the linear operator $K_1\varphi$. Let $p_1(x), \dots, p_l(x)$ be the corresponding eigenfunctions, and $q_1(x), \dots, q_l(x)$ their adjoint functions. In what follows we assume that the systems $\{p_i\}_1^l$ and $\{q_i\}_1^l$ can be normalized so that

$$(p_i, q_j) = \delta_{ij}. \tag{3}$$

Introduce the operator $P\varphi$ by the equality

$$P\varphi = \sum_{i=1}^{l} (\varphi, q_i) p_i. \tag{4}$$

The operator $P\varphi$ projects the space C onto the subspace E_l, consisting of the eigenfunctions of K_1.

Let us further suppose that the operators $K_n\varphi^n, K_{n+1}\varphi^{n+1}, \dots$ satisfy the following conditions:

$$PK_n\varphi^n \equiv PK_{n+1}\varphi^{n+1} \equiv \dots \equiv PK_{r-1}\varphi^{r-1} \equiv 0 \quad [(\varphi \in E_l)]; \tag{5}$$

$$PK_r\varphi^r \neq 0 \quad (\varphi \in E_l,\ \varphi \neq 0), \tag{6}$$

where

$$n \leq r < 2n - 1 \tag{7}$$

(for $r = n$ condition (5) drops out).

Introduce the operator

$$K\varphi = K_1\varphi + K_n\varphi^n + K_{n+1}\varphi^{n+1} + \dots \tag{8}$$

Theorem 1. *Under conditions (5), (6), (7), the point one is a bifurcation point of the operator K.*

For the proof it suffices to note that the index of the zero solution of the equation

$$\varphi = K\varphi \tag{9}$$

is equal to r^l (see ([6], [7])).

2. Theorem 1 shows that for λ close to one, equation (1) has small nonzero solutions. Let us proceed to find some of these solutions in the form of series in fractional powers of $\lambda - 1$.

Denote $(\lambda - 1)^{\frac{1}{2r-2}} = \mu$ and write equation (1) in the form

$$\varphi = (1 + \mu^{2r-2})(K_1\varphi + K_n\varphi^n) + \ldots + K_r\varphi^r + \ldots). \tag{10}$$

Let

$$\varphi(x) = \mu^2\varphi_2(x) + \mu^3\varphi_3(x) + \ldots \tag{11}$$

be a solution of equation (10). Substituting the series (11) into (10) and comparing coefficients of equal powers of μ, we obtain a system of equations. These equations have the form

$$\varphi_i = K_1\varphi_i + F_i(\varphi_2, \varphi_3, \ldots, \varphi_{i-(2n-2)}) \quad (\text{for } i < 2n \quad F_i = 0). \tag{12}$$

The i-th equation (12), if it is solvable, has a solution depending linearly on l arbitrary constants $c_{1i}, \ldots, c_{li}$. The values of these arbitrary constants are determined from the solvability conditions of the subsequent equations (12). Thus, to determine the constants $\{c_{j2}\}_1^l$, from the solvability condition of the $2r$-th equation (12) we obtain the system

$$\left(K_r\left(\sum_{j=1}^{l} c_{j2}p_j\right)^r,\ q_s\right) + c_{s2} = 0 \quad (s = 1, 2, \ldots, l). \tag{13}$$

The system (13) can be written in the form

$$\Phi(z) = 0, \tag{14}$$

where Φ is some nonlinear mapping in the l-dimensional complex Euclidean space R_l, and z is a point of the same space. The set M of solutions of equation (14) is bounded by virtue of (6) and, consequently, has dimension zero ([8]). Let z_0 be some simple point of the set M. In this case the following constants $\{c_{jk}\}_1^l$ $(k \geq 3)$ are determined from linear systems with one and the same determinant, not equal to zero. Thus, to a simple point of the set M there corresponds one solution of equation (10). This solution is representable in the form of the series (11), and the process of determ–

of the coefficients of which stabilizes(9). Let us note that if unity is a simple eigenvalue of the kernel $K_1(x, s)$, then the manifold M consists only of simple points.

Suppose now that z_0 is a multiple point of the manifold M. Expand the operator $\Phi(z_0 + h)$ in a Taylor series in h:

$$\Phi(z_0 + h) = A_1h + A_2h^2 + \ldots + A^r_{rh}. \tag{15}$$

Denote by R_m $(m < l)$ the m-dimensional eigensubspace of the operator A_1 corresponding to the zero eigenvalue. As in item 1, introduce the operator P_1, projecting R_l onto R_m (here we assume that a condition analogous to (3) is satisfied). Suppose that

$$P_1 A_2 h^2 \neq 0 \qquad (h \in R_m,\ h \neq 0). \tag{16}$$

From the solvability conditions of the $(2r+1)$-st and $(2r+2)$-nd equations (12) it follows that the points $h = (c_{13}, c_{23}, \dots, c_{l3})$ belong to R_m and satisfy the equation

$$P_1 A_2 h^2 = f, \tag{17}$$

where f is a certain definite element of the space R_m. The solutions of equation (17) form a certain zero-dimensional manifold N in R_m. Suppose further that all points of the manifold N are simple, i.e., for any point $h \in N$

$$P_1 A_2 h g \neq 0 \qquad (g \in R_m,\ g \neq 0). \tag{18}$$

To determine $\{c_{jk}\}_1^l$ for $k \geq 4$, from the solvability condition of the $(2r+k-2)$-nd equation (12) we obtain a linear system (L_k) with determinant equal to zero. From the system (L_k), under the condition of its solvability, the vector $(c_{1k}, c_{2k}, \dots, c_{lk})$ is determined up to an addend $h_k \in R_m$. The vector h_k is determined uniquely from the solvability condition of the system (L_{k+1}). The described process of determining the coefficient functions differs from a stabilizing process, but allows one to construct, in the form of the series (11), a formal solution of equation (10). It can be proved that this formal solution is an actual solution by applying, for example, to equation (10) the implicit-function theorem according to the scheme set forth in (10).

By virtue of (16) and (18), the number of solutions of the form (11) of equation (10) corresponding to any stationary point z_0 of the field $\Phi(z)$ coincides with the index of this point. Consequently, the total number of solutions of the form (11) of equation (10) coincides with the degree r^l of the field $\Phi(z)$ on a sufficiently large sphere. On the other hand, it can be shown that under the assumptions we have made all small solutions of equation (1), representable in fractional powers of $\lambda - 1$, have the form (11). We formulate the result obtained.

Theorem 2. *Let the conditions (5), (6), (7) be satisfied for equation (1). Let the conditions (16), (18) be satisfied for each multiple point of the solution manifold of the system (13).*

Then, for λ close to unity, equation (1) has $r^l - 1$ nonzero solutions expandable in series in positive fractional powers of $\lambda - 1$.

3. We give theorems on the indices of solutions of equation (1).

Theorem 3. *The index of any isolated solution φ ($\|\varphi\| < R$) of equation (1) is positive.*

Theorem 4. *Let the conditions of Theorem 2 be satisfied.*

Then, for sufficiently small nonzero μ, the index of any solution (11) of equation (10) is equal to unity.

4. Since the index of the zero solution of the equation $\varphi = K\varphi$ is equal to r^l, the following proposition follows from Theorems 2, 3, and 4.

Theorem 5. *Let the conditions of Theorem 2 be satisfied. Let there exist positive numbers ε and δ such that, for $|\lambda - 1| < \varepsilon$, all solutions φ of equation (1) satisfying the inequality $\|\varphi\| < \delta$ are isolated.*

Then there exist positive numbers ε_1 and δ_1 such that, for $|\lambda - 1| < \varepsilon_1$, $\lambda \neq 1$, equation (1) has, inside the ball $\|\varphi\| < \delta_1$, exactly r^l of the solutions described above and has no other solutions.

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Note: Figure translations are in progress. See original paper for figures.