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Abstract

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MATHEMATICS

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ON SOME ESTIMATES CONNECTED WITH INTEGRAL OPERATORS AND SOLUTIONS OF ELLIPTIC EQUATIONS

(Presented by Academician S. L. Sobolev on 28 I 1961)

In a number of works of recent years ([1,2]), a priori estimates in L_p have been derived for the highest derivatives of solutions of various boundary-value problems for elliptic equations (and systems). From these results it follows, for example, that the second derivatives of the solution of the first boundary-value problem for the Poisson equation in a sufficiently regular domain Ω of n-dimensional space E_n with boundary S

$$\Delta u = f, \qquad u\big|_S = 0 \tag{1}$$

belong to L_p, if $f \in L_p \left(p > \frac{2n}{n+2}\right)$, and the estimate

$$\|D^2 u\|_{L_p(\Omega)} \le K_p \|f\|_{L_p(\Omega)} \tag{2}$$

holds, where K_p is a constant depending only on Ω and on p; $D^r u$ denotes any derivative of u with respect to $x_1, \dots, x_n$ of order r.

If Ω is a bounded domain and f is a bounded function, then, of course, estimate (2) holds for arbitrarily large p. However, as simple examples show, for $p = \infty$ estimate (2) is impossible (the second derivatives are unbounded).

Meanwhile, in solving certain nonlinear problems ([3]), it proves necessary to determine how the differential properties of the solution improve in this case. Further, it is known ([4]) that if the right-hand side $f(x)$ of equation (1) belongs to the space $B^{0,\lambda}$*, and the boundary S is sufficiently smooth, then the second derivatives of u also belong to $B^{0,\lambda}$, and an estimate analogous to (2), with some constant K_λ, is valid. For $\lambda = 0$ (discontinuous right-hand side) and $\lambda = 1$ (Lipschitz condition), these facts no longer hold, and the question again arises as to what the differential properties of the solution are.

The method used in the present work is connected with the study of the growth of the constant K_p in (2) as $p \to \infty$ (respectively, K_λ as $\lambda \to 1$ in the case of Hölder norms).

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The same method is then applied to refine theorems on integrals of potential type and embedding theorems ([5]) in certain critical cases.

The results obtained here find applications in proving uniqueness theorems and in investigating the differential properties of solutions of nonlinear problems.

* $B^{k,\lambda}$ is the space of functions defined in the domain Ω and having all derivatives up to order k, satisfying a Hölder condition with exponent λ; the norm in $B^{k,\lambda}$ is equal to the sum of the maxima of all derivatives of orders $0, 1, \dots, k$ and their Hölder constants.

1. Consider the first boundary-value problem for a second-order elliptic equation in a bounded domain Ω with boundary S:

$$\sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x); \tag{3}$$

$$u\big|_S = 0. \tag{4}$$

Theorem 1. Let $S \in C^{r+2}$; let $b_i(x)$, $c(x)$ have bounded generalized derivatives of order r; and let $a_{ik}(x) \in B^{r,\mu}$ $(0 < \mu < 1)$.* Then for the generalized solution of class $W_p^{(r+2)}$ $(r \geq 0)$ of problem (3), (4) the estimate holds $(p \geq p_0 > 1)$

$$\|u\|_{W_p^{(r+2)}} \leq c \left(p \|f\|_{W_p^{(r)}} + p^{\frac{1-\alpha}{\mu}} \|u\|_{B^{r-1,\alpha}} \right), \tag{5}$$

where $0 < \alpha \leq 1$ is arbitrary, and c does not depend on p.

The proof of Theorem 1 is based on the following result of A. P. Calderón and A. Zygmund ([7]). The singular integral

$$Sf = \int_{E_n} \frac{K\left(\dfrac{x-y}{|x-y|}\right)}{|x-y|^n} f(y)\, dy, \tag{6}$$

where the function $K(\theta)$ satisfies the conditions: 1) $\int_{|\theta|=1} K(\theta)\, d\theta = 0$; 2) $|K(\theta_1) - K(\theta_2)| < \omega(|\theta_1 - \theta_2|)$, where $\omega(t)$ is a continuous increasing function such that $\omega(t) \geq t$, $\int_0^1 \frac{\omega(t)}{t}\, dt < \infty$, defines an operator from L_p to L_p $(p > 1)$, and the estimate holds** $(p \geq p_0 > 1)$

$$\|Sf\|_{L_p(E_n)} \leq cp \|f\|_{L_p(E_n)}. \tag{7}$$

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Let us note that inequalities (5) and (7) are sharp in the sense of the growth of the constants as $p \to \infty$, as is shown by the example of problem (1) in the case of the n-dimensional ball $|x| \le 1$, with $f = (n+2)x_1 x_2/|x|^2$. The corresponding solution is $u_0(x) = x_1 x_2 \ln|x|$. In this case $\partial^2 u_0/\partial x_1 \partial x_2 = \ln|x| +$ a bounded function, and

$$\left\| \frac{\partial^2 u_0}{\partial x_1 \partial x_2} \right\|_{L_p} \sim \frac{1}{ne} p \quad \text{as } p \to \infty.$$

Consequently, for example, for the constants K_p in (2) we have: $c_1 p \le K_p \le cp$.

In many cases (for example, for self-adjoint semibounded problems) the second term on the right-hand side of (5) can be majorized by $c\|f\|_{L_{p_0}}$ $(p_0 \ge n/\alpha)$. Then inequality (5) can be put in the form

$$\|u\|_{W_p^{(r+2)}} \le Cp\|f\|_{W_p^{(r)}}. \tag{8}$$

Corollary 1. If $f \in W_\infty^{(r)}$, i.e. has bounded generalized derivatives of order r, and (8) holds, then there exists a constant $\gamma > 0$ such that

$$\int_\Omega e^{\beta|D^{r+2}u|}\, dx < \infty, \tag{9}$$

provided only that $\beta\|f\|_{W_\infty^{(r)}} < \gamma$.

* For the definitions of the classes C^k, $A_{k,\lambda}$, see (4,6).

** The symbols c, c_i everywhere denote certain constants independent of p (or, respectively, of λ).

(9) is obtained immediately if the integrand is expanded in a Taylor series in β, and then (8) and Stirling' s formula

$$k! \sim \sqrt{2k\pi}\, K^k e^{-k}$$

are applied.

Corollary 2. If the conditions of Theorem 1 and Corollary 1 are satisfied and, moreover, $S \in \Lambda_{r+2,\lambda}$, $(0 < \lambda < 1)$, while $f \in C^{(r)}$, i.e., has continuous derivatives of order r, then (9) holds for every $\beta > 0$.

For the proof, approximate f in $C^{(r)}$ by functions $f^* \in B^{r,\lambda}$. As is known, the corresponding solution u^* of problem (3), (4) satisfies $u^*(x) \in B^{r+2,\lambda}$. For every $\rho > 0$ we have

$$\int_\Omega e^{\beta|D^{r+2}u|}\, dx \le e^{\beta\|u^*\|_{C^{(r+2)}}} \int_\Omega e^{\beta|D^{(r+2)}(u-u^*)|}\, dx < \infty \tag{10}$$

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by Corollary 1, if $B\|f-f^*\|_{C^{(r)}} \leq \gamma$.

It is proved similarly that if $\hat{f}(x)$ in (6) is a finite bounded function, then for every bounded domain Ω' there exists a constant γ_0 such that

$$\int_{\Omega'} e^{\beta|Sf|}\, dx < \infty. \tag{11}$$

provided only that

$$\beta \text{ Vrai} \sup_{x\in E_n} |f(x)| < \gamma_0.$$

If, however, $f(x)$ is continuous in E_n, then (11) is valid for every $\beta > 0$. For $n = 1$ the latter is the content of a well-known theorem of V. I. Smirnov [(8)], proved by methods of the theory of analytic functions.

Theorem 2. Let $a_{ik}(x), b_i(x), c(x) \in B^{r,1}$, $S \in \Lambda_{r+2,1}$. Then, for λ close to 1, the estimate

$$\|u\|_{B^{r+2,\lambda}} \leq \frac{c}{1-\lambda}\left(\|f\|_{B^{r,\lambda}} + \|u\|_{B^{1,1}}\right) \tag{12}$$

is valid.

For those cases in which an estimate of the lower derivatives is known, (12) takes the form

$$\|u\|_{B^{r+2,\lambda}} \leq \frac{c}{1-\lambda}\|f\|_{B^{r,\lambda}}. \tag{13}$$

In the latter case the estimate

$$z_{12} = |D^{r+2}u(x^1) - D^{r+2}u(x^2)| \leq c_1 r(1+|\ln r|), \qquad r = |x^1 - x^2|. \tag{14}$$

is valid. It is enough to prove (14) in the case $r < 1$. Rewrite (14) in the form

$$e^{z_{12}/c_1 r} \leq \frac{e}{r}. \tag{15}$$

Expand the left-hand side of the last equality in a Taylor series. Applying inequality (13) with $\lambda = 1 - 1/k$, we obtain

$$e^{z_{12}/c_1 r} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{z_{12}^k}{c_1^k r^k} \leq \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{c_1^k}\, c^k k^k \frac{1}{r}. \tag{16}$$

(14) follows from (15), (16), if one observes that the series on the right-hand side of (16) will converge for $c_1 > ec$, and, as $c_1 \to \infty$, will tend to 1.

Theorems 1 and 2 admit a generalization to equations of higher order and to systems.

2. In various inequalities of embedding type there often occurs a situation in which some of the constants entering these inequalities tend to infinity as a certain parameter approaches its critical value, and the inequalities themselves lose meaning.

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The method applied above makes it possible to investigate such cases simply. We shall give examples of facts obtained in this way.

Theorem 3. Suppose

$$u(x) = \int\limits_{\Omega} \frac{f(y)}{|x-y|^{\alpha}}\,dy; \tag{17}$$

Ω is a bounded domain E_n; $f \in L_{\frac{n}{n-\alpha}}$. Then the relations

$$\|u\|_{L^{q^*}(\Omega_s)} \le c(q^*)(q^*)^{\alpha/n}\|f\|_{L_{\frac{n}{n-\alpha}}}; \tag{18}$$

$$\int\limits_{\Omega_s} e^{\gamma|u(x)|^{n/\alpha}}\,dx < \infty, \tag{19}$$

hold, where Ω_s is the section of Ω by an s-dimensional hyperplane; $0 < \alpha < n$; $\gamma > 0$ is arbitrary; $c(q^*)$ is bounded as $q^* \to \infty$.

Using the fact that the left-hand side of (19) is an entire function of γ, we show that, for a fixed function f, $c(q^*) \to 0$.

(18) and (19), where formally $\alpha = n$ is put, are valid for singular integrals (6) with continuous f. We note that (18) is obtained as a "by-product" in the proof of the theorem on potentials in (5).

Theorem 4. If in (17) $0 < \alpha \le n-1$, then, as $\lambda \to 1$, the inequalities ($r = |x^1 - x^2|$)

$$|u(x^1) - u(x^2)| \le c(1-\lambda)^{-\frac{\alpha+\lambda}{n}}\|f\|_{L_{\frac{n}{n-\alpha-\lambda}}}\, r^{\lambda}; \tag{20}$$

$$|u(x^1) - u(x^2)| \le c_1 r\left(1 + |\ln r|^{\frac{\alpha+1}{n}}\right). \tag{21}$$

are valid.

The applications to embedding theorems are obvious.

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