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Abstract

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SEMIEUCLIDEAN AND SEMINONEUCLIDEAN SPACES

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Semieuclidean and seminoneuclidean spaces were first considered by Sommerville ([1]) and, under the name "semieuclidean" and "seminoneuclidean," by B. A. Rosenfeld ([2], pp. 49 and 152). Up to now only special cases of semieuclidean and seminoneuclidean spaces have been studied ([3–15]); the present paper is devoted to the general theory of these spaces.

1. **Semieuclidean spaces.** Define the **semieuclidean space** ${}^{l_1 \dots l_r}R_n^{m_1 \dots m_{r-1}}$ as the space A_n in which r scalar products are given

$$(xy)_a = \sum \varepsilon_a^i x_a^i y_a^i, \tag{1}$$

where $0 = m_0 < m_1 < m_2 < \dots < m_r = n$, $a = 1, 2, \dots, r$, $i_a = m_{a-1} + 1, \dots, m_a$, $\varepsilon_a^i = \pm 1$, and -1 occurs among the numbers ε_a^i exactly l_a times. The product $(xy)_a$ is defined for those vectors for which all coordinates x^i with $i \le m_{a-1}$ are zero. For such vectors there is also defined the **modulus** $|x|_a = \sqrt{(xx)_a}$, which is a nonnegative number when $(xx)_a \ge 0$ and a number from the upper half-plane of the complex variable when $(xx)_a < 0$. When the points $A(x)$ and $B(y)$ are such that the modulus $|y - x|_a$ is defined for the vector $y - x$, we call this modulus the **distance** d_a between A and B, and call the line AB a line of ***a*-th order**. When the scalar product $(xy)_a$ is defined for the vectors x and y, and the moduli $|x|_a$ and $|y|_a$ are nonzero, we define the angle between them as the real or complex number determined by the relation

$$\cos \varphi_a = (xy)_a / |x|_a |y|_a. \tag{2}$$

We shall call a **hypersphere** of the space ${}^{l_1 \dots l_r}R_n^{m_1 \dots m_{r-1}}$ with center at the point $A(a)$ and radius R the geometric locus of points whose first distance from the point A is equal to R. A hypersphere with center at the point 0 and radius 1 has the equation $(xx)_1 = 1$. This hypersphere is a cylinder with $(n - m_1)$-dimensional planar generators, which are the spaces ${}^{l_2 \dots l_r}R_{n-m_1}^{m_2 - m_1 \dots m_{r-1} - m_1}$. Points $A(x)$ and $B(y)$, belonging to different planar generators, together with the point 0 determine a Euclidean plane that cuts circles from the hypersphere.

Therefore, for the distance between them, measured on the hypersphere, it is natural to take the angle φ_1 between the vectors x and y; (2) can be rewritten in the form $\cos\varphi_1 = (xy)_1$.

If the space A_n is extended to the projective space P_n, all planar generators of the hypersphere are extended by a common $(n - m_1 - 1)$-

-dimensional infinitely distant plane, and the hypersphere will turn into a cone with an $(n - m_1 - 1)$-dimensional vertex plane.

2. Semineuclidean spaces. Define the **semineuclidean space** ${}^{l_0 l_1 \dots l_r}S_n^{m_0 m_1 \dots m_{r-1}}$ as the hypersphere in the space ${}^{l_0 \dots l_r}R_{n+1}^{m_0 \dots m_{r-1}}$ with diametrically opposite points identified; the geometry of the space ${}^{l_0 \dots l_r}R_{n+1}^{m_0 \dots m_{r-1}}$ is determined by the scalar products (1), where $0 \le m_0 < m_1 < \dots < m_r = n$. In this notation the equation of the hypersphere takes the form $(xx)_0 = 1$. If we adjoin to the hypersphere the infinitely distant points of its plane generators, the space ${}^{l_0 \dots l_r}S_n^{m_0 \dots m_{r-1}}$ may be regarded as a metrized projective space P_n. The infinitely distant points of the hypersphere form in the space P_n a second-order hypercone $(xx)_0 = 0$, called the **absolute hypercone** of the semineuclidean space; the infinitely distant points of the plane generators of the hypersphere form the vertex plane of this hypercone. In this plane there is defined a second-order cone $(xx)_1 = 0$, in its vertex plane a second-order cone $(xx)_2 = 0$, etc., and in the vertex plane of the second-order cone $(xx)_{r-1} = 0$ there is defined a nondegenerate quadric $(xx)_r = 0$; in the case when the cone $(xx)_{r-1} = 0$ has a point vertex, the quadric $(xx)_r = 0$ is a pair of coincident points, coinciding with this vertex. The cone $(xx)_a = 0$ will be called the a-th **absolute cone** of the space, and the vertex plane of the a-th absolute cone the a-th **absolute plane** of the space; the totality of all absolute cones and absolute planes of the space will be called the **absolute** of the space.

If the points $A(x)$ and $B(y)$ lie on the $(a-1)$-st absolute plane but do not lie on the a-th, their projective coordinates may be normalized by the condition

$$(xx)_a = 1. \tag{3}$$

If these points do not lie on one plane generator of the hypersphere (3) in the $(a-1)$-st absolute plane, then as the distance δ_a between them we shall take the distance between the corresponding points of this hypersphere, measured on this hypersphere, which is determined by the relation $\cos\delta_a = (xy)_a$. In this case the line AB is a noneuclidean line; we shall call the distance δ_a the **noneuclidean distance of the a-th order**, and the line AB a **noneuclidean line of the a-th order**. If, however, these points lie on one plane generator of the hypersphere (3), but are not its infinitely distant points, then $\delta_a = 0$, and between these points there is defined the distance $d_{a+1} = |y - x|_{a+1}$; the line AB is a euclidean line; we shall call the distance d_a a **euclidean distance of the a-th order**, and the line AB a **euclidean line of the a-th order**.

The semieuclidean space ${}^{l_1\dots l_r}R_n^{m_1\dots m_{r-1}}$ may be regarded as the semineuclidean space ${}^{0l_1\dots l_r}S_n^{0m_1\dots m_{r-1}}$ without its absolute hyperplane, with which the absolute hypercone of this space coincides.

3. Angles between hyperplanes. To any two hyperplanes of the space ${}^{l_0\dots l_r}S_n^{m_0\dots m_{r-1}}$ there corresponds a number equal to the distance between the points of the dual space ${}^{l_r l_{r-1}\dots l_0}S_n^{n-m_{r-1}-1\dots n-m_0-1}$, whose projective coordinates are numerically equal to the tangential coordinates of these hyperplanes; we shall call these numbers the **angles between the hyperplanes**. Since semieuclidean spaces, supplemented by an infinitely distant hyperplane, are dual-

semieuclidean spaces; this definition of angles between hyperplanes also applies to semieuclidean spaces.

If the hyperplanes $\alpha(p)$ and $\beta(q)$ pass through the $(a+1)$-st absolute plane, but do not pass through the a-th absolute plane, their tangential coordinates may be normalized by the condition $(pp)_a = 1$. In this case the angle φ_a between the hyperplanes is determined by the relation $\cos\varphi_a = (pq)_a$, or, if $\varphi_a = 0$, the angle $f_a = |q-p|_{a-1}$ is determined between them. We shall call the angles φ_a and f_a, respectively, the non-Euclidean and Euclidean angles of a-th order, and the pencils of hyperplanes containing the hyperplanes α and β, respectively, the non-Euclidean and Euclidean pencils of a-th order. In particular, in the case of the Euclidean space ${}^lR_n$, supplemented by an infinitely distant hyperplane, φ_1 and f_1 are the angle between intersecting hyperplanes and the distance between parallel hyperplanes, while the non-Euclidean and Euclidean pencils of first order are the pencil of intersecting hyperplanes and the pencil of parallel hyperplanes.

4. **Motions.** We shall call a **motion** of the semieuclidean space

$$ {}^{l_1\dots l_r}R_n^{m_1\dots m_{r-1}} $$

an affine transformation of the space A_n preserving all distances d_a. Since the modulus $|x|_a$ is defined only for those vectors x for which $x^i = 0$ when $i < m_{a-1}$, a motion preserving the modulus $|x|_a$ takes a vector for which $x^i = 0$ when $i < m_{a-1}$ into a vector having the same property.

Requiring that the affine transformation

$$ {}'x^i = \sum_j A^i_j x^j $$

take vectors for which $x^i = 0$ when $i < m_{a-1}$ into vectors having the same property, and preserve the distance d_a, we obtain that on the main diagonal of the matrix (A^i_j) there stand orthogonal matrices of order $(m_a - m_{a-1})$ and of index l_a ((2), p. 56); all elements above and to the right of these matrices are equal to zero, while no conditions are imposed on the elements below and to the left of these matrices. We shall call the matrix of the transformation

$$'x^i = \sum_j A^i_j x^j$$

a **semiorthogonal matrix of indices** $l_1, l_2, \ldots, l_r$. We shall call a **motion** of the semieuclidean space

$$^{l_0 \ldots l_r}S_n^{m_0 \ldots m_{r-1}}$$

a transformation of this space determined by a motion of the space

$$^{l_0 \ldots l_r}R_{n+1}^{m_0 \ldots m_{r-1}},$$

which carries into itself the hypersphere on which the geometry of the space

$$^{l_0 \ldots l_r}S_n^{m_0, m_1 \ldots m_{r-1}}$$

is realized. Therefore the motion of the space

$$^{l_0 \ldots l_r}S_n^{m_0 \ldots m_{r-1}}$$

is expressed by the formula

$$'x^{i_0} = \sum_{j_0} U^{i_0}_{j_0} x^{j_0}, \qquad 'x^{i_1} = \sum_{j_0} T^{i_1}_{j_0} x^{j_0} + \sum_{j_1} U^{i_1}_{j_1} x^{j_1}, \ldots$$

$$'x^{i_r} = \sum_{j_0} T^{i_r}_{j_0} x^{j_0} + \sum_{j_1} T^{i_r}_{j_1} x^{j_1} + \cdots + \sum_{j_{r-1}} T^{i_r}_{j_{r-1}} x^{j_{r-1}} + \sum_{j_r} U^{i_r}_{j_r} x^{j_r}. \qquad (4)$$

Since semiorthogonal matrices depend on the same number of parameters as orthogonal matrices of the same order, the motion matrices (4) of the space

$$^{l_0 \ldots l_r}S_n^{m_0 \ldots m_{r-1}}$$

depend on the same number of parameters ($n(n+1)/2$) as do the motions of the spaces ^lR_n and ^lS_n.

Under a motion, all distances δ_a and d_a between points, and the angles φ_a and f_a between hyperplanes, remain unchanged.

Every one-to-one transformation of the spaces

$$^{l_1 \ldots l_r}R_n^{m_1 \ldots m_{r-1}}$$

and

$$^{l_0 \ldots l_r}S_n^{m_0 \ldots m_{r-1}},$$

preserving distances between points of these spaces, is a motion; this is proved in exactly the same way as the analogous theorems in the spaces ^lR_n and ^lS_n.

5. **Conformal transformations.** Consider the semihyperbolic space

$$ {}^{l_0 \dots l_r} S_n^{m_0 \dots m_{r-1}} $$

with $m_0 > 0$. We shall call, for $n = 2$, a **cycle**, and for $n > 2$, a **hypercycle** of such a space ···

of the space is the geometric locus of points which, when it is represented as a hypersphere of a semi-Euclidean space, is represented by a section by a hyperplane. A hyperplane of general form in the space A_{n+1} is defined by the equation $\sum_i p_i x^i = p$; therefore the coordinates of the common points of this hyperplane and the hypersphere $(xx)_0 = 1$ satisfy the equation

$$ p^2 \left(\sum_{i_0} \varepsilon_{i_0} x^{i_0\,2} \right) = \left(\sum_i p_i x^{i_0} \right)^2 . \tag{5} $$

Equation (5) is also the equation of a hypercycle of the space ${}^{l_0 \dots l_r} S_n^{m_0 \dots m_{r-1}}$. It shows that a hypercycle is a special case of a quadric of the space P_n. A hypersphere of the space ${}^{l_0 \dots l_r} S_n^{m_0 \dots m_{r-1}}$, i.e. the geometric locus of points whose zero distance from $A(a)$ is equal to R, has the equation

$$ \cos^2 R \left(\sum \varepsilon_{i_0} x^{i_0\,2} \right) = \left(\sum \varepsilon_{i_0} p_{i_0} x^{i_0} \right)^2 , $$

i.e. it is a special case of a hypercycle.

At each point $M_0(x_0)$ of the hypercycle (5) one can define the tangent hyperplane

$$ p^2 \left(\sum_{i_0} \varepsilon_{i_0} x_0^{i_0} x^{i_0} \right) = \sum_i p_i x_0^i \sum_i p_i x^i . $$

Let us complete the space A_{n+1} to the projective space P_{n+1} by adding the hyperplane $x^{-1} = 0$. We introduce into this space P_{n+1} the metric of the space

$$ {}^{l_0+1, l_1 \dots l_r} S_{n+1}^{m_0+1, m_1+1 \dots m_{r-1}+1} , $$

taking as the zero absolute quadric the hypersphere of the space ${}^{n-m_0} R_{n+1}$ which represents the space ${}^{l_0 \dots l_r} S_n^{m_0 \dots m_{r-1}}$, and taking as the remaining absolute quadrics the remaining absolute quadrics of the space ${}^{l_0 \dots l_r} S_n^{m_0 \dots m_{r-1}}$. Then the angles of order a between hyperplanes in the space

$$ {}^{l_0+1 \dots l_r} S_{n+1}^{m_0+1 \dots m_{r-1}+1} $$

and the tangent hyperplanes to hypercycles at their point of intersection in the space ${}^{l_0 \dots l_r}S_n^{m_0 \dots m_{r-1}}$ are equal. Since the angles between tangent hyperplanes in all cases do not depend on the point of intersection, we shall call these angles the **angles between hypercycles**. Therefore, the manifold of hyperplanes of the space

$$ {}^{l_0+1 \dots l_r}S_{n+1}^{m_0+1 \dots m_{r-1}+1} $$

for $m_0 > 0$, if the distance between hyperplanes is taken to be the angle between them, is isometric to the manifold of hypercycles of the space

$$ {}^{l_0 \dots l_r}S_n^{m_0 \dots m_{r-1}} $$

for $m_0 > 0$, if the distance between hypercycles is also taken to be the angle between them. On the other hand, the manifold of planes of the space

$$ {}^{l_0+1 \dots l_r}S_{n+1}^{m_0+1 \dots m_{r-1}+1} $$

is isometric to its dual space

$$ {}^{l_r \dots l_0+1}S_{n+1}^{n-m_{r-1} \dots n-m_0}. $$

The motions of the space

$$ {}^{l_r \dots l_0+1}S_{n+1}^{m_{r-1}+1 \dots m_0+1} $$

represent transformations of the space

$$ {}^{l_0 \dots l_r}S_n^{m_0 \dots m_{r-1}}, $$

which carry hypercycles into hypercycles while preserving the angles between them. By analogy with Euclidean and non-Euclidean spaces, they may be called **conformal transformations**.

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Note: Figure translations are in progress. See original paper for figures.