THE BOUNDARY-VALUE PROBLEM FOR THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A SPHERICALLY LAYERED ANISOTROPIC DISSIPATIVE MEDIUM

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Abstract

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THE BOUNDARY-VALUE PROBLEM FOR THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A SPHERICALLY LAYERED ANISOTROPIC DISSIPATIVE MEDIUM

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In paper ([1]) a method of normal waves was published. Its formulation was based on the spectral theory of self-adjoint linear operators, as a result of which it was suitable for solving a comparatively narrow class of boundary-value problems on wave propagation in layered media without losses. The spectral theory of linear non-self-adjoint operators developed in recent decades makes it possible to extend the application of the method of normal waves ([1]) to layered media with losses and, naturally, to take radiation into account. Here a sufficiently general case of such media is considered, having a direct relation to the spherical semiconducting Earth surrounded by a magneto-anisotropic ionosphere. It includes Watson' s problems ([2]) and their analogues from acoustics and seismology.

1. We seek a stationary electromagnetic field of frequency ω in a spherical coordinate system r, θ, φ, produced by current densities

$$I_r(r,\theta)e^{-i\omega t}; \qquad I_\theta = 0; \qquad I_\varphi = 0. \tag{1}$$

The medium is determined by the tensor of dielectric constant

$$\|\varepsilon^k\| = \begin{Vmatrix} \varepsilon^k_{rr} & 0 & 0 \\ 0 & \varepsilon^k_{\theta\theta} & \varepsilon^k_{\theta\varphi} \\ 0 & -\varepsilon^k_{\theta\varphi} & \varepsilon^k_{\varphi\varphi} \end{Vmatrix}, \qquad k = 0, 1, 2, \dots, N. \tag{2}$$

Its components are complex functions of r, continuous on the intervals $r_{k-1} \leqslant r \leqslant r_k$ and differentiable. As $r \to \infty$, ε_{rr}, $\varepsilon_{\theta\theta}$, and $\varepsilon_{\varphi\varphi}$ tend adiabatically slowly to $1 + i\Delta$, where Δ is a small but finite positive quantity, while $\varepsilon_{\theta\varphi} \to 0$. The magnetic permeability is $\mu = 1$.

The field components satisfy Maxwell' s equations:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta H^k_\varphi\right) = -\frac{i\omega}{c}\varepsilon^k_{rr}E^k_r + \frac{4\pi}{c}I_r; \qquad \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E^k_\varphi\right) = \frac{i\omega}{c}H^k_r,$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rH_\varphi^k\right) = \frac{i\omega}{c}\left[\varepsilon_{\theta\theta}^k E_\theta^k + \varepsilon_{\theta\varphi}^k E_\varphi^k\right]; \qquad -\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\varphi^k\right) = \frac{i\omega}{c}H_\theta^k, \tag{3}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rH_\theta^k\right) - \frac{\partial H_r^k}{\partial \theta}\right] = \frac{i\omega}{c}\left[\varepsilon_{\theta\varphi}^k E_\theta - \varepsilon_{\varphi\varphi}^k E_\varphi^k\right];$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\theta^k\right) - \frac{\partial E_r^k}{\partial \theta}\right] = \frac{i\omega}{c}H_\varphi^k.$$

At the boundaries of the layers $r = r_k$ the continuity conditions hold for the tangential components E_θ, E_φ, H_θ, and H_φ; at the point $r_{-1} = 0$ the boundedness conditions hold: $\operatorname{mod} E < M$, $\operatorname{mod} H < M$, while as $r \to \infty$, $\operatorname{mod} E$ and $\operatorname{mod} H \to 0$.

2. **Formulation of the problem in the language of operators.** Introduce the vector function

$$\begin{vmatrix} B(r,\theta) \\ A(r,\theta) \end{vmatrix}, \qquad \text{where} \qquad E_\varphi = \frac{1}{r}\frac{\partial A}{\partial \theta}; \qquad H_\varphi = \frac{1}{r}\frac{\partial B}{\partial \theta}. \tag{4}$$

Then equations (3) are written in operator-matrix form:

$$l_r^{(k)}\begin{vmatrix} B_k \\ A_k \end{vmatrix} + l_\theta^{(k)}\begin{vmatrix} B_k \\ A_k \end{vmatrix} = \frac{4\pi}{c}r^2\begin{vmatrix} I_r \\ 0 \end{vmatrix}, \qquad k = 0, 1, 2, \ldots, N,$$

where

$$l_r^{(k)} = \begin{vmatrix} \varepsilon_{rr}^k r^2 \dfrac{\partial}{\partial r}\left[\dfrac{1}{\varepsilon_{\theta\theta}^k}\cdot\right] + k_0^2\varepsilon_{rr}^k r^2 \cdot; & -ik_0\varepsilon_{rr}^k r^2 \dfrac{\partial}{\partial r}\left[\dfrac{\varepsilon_{\theta\varphi}^k}{\varepsilon_{\theta\theta}^k}\cdot\right] \\ \dfrac{ik_0\varepsilon_{\theta\varphi}^k r^2}{\varepsilon_{\theta\theta}^k}\dfrac{\partial}{\partial r}\cdot; & r^2\dfrac{\partial^2}{\partial r^2}\cdot + k_0^2 r^2\left[\varepsilon_{\varphi\varphi}^k + \dfrac{(\varepsilon_{\theta\varphi}^k)^2}{\varepsilon_{\theta\theta}^k}\right]\cdot \end{vmatrix}, \tag{5}$$

$$l_\theta^{(k)} = \begin{vmatrix} \mathcal{L}; & 0 \\ 0; & \mathcal{L} \end{vmatrix}, \qquad \text{where } \mathcal{L} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\cdot\right). \tag{6}$$

The conditions at $r = r_k$, $r = 0$, and $r \to \infty$ take the form

$$\Delta^{(k)}\begin{vmatrix} B_k \\ A_k \end{vmatrix}_{r=r_k} = \Delta^{(k+1)}\begin{vmatrix} B_{k+1} \\ A_{k+1} \end{vmatrix}_{r=r_k}; \qquad \begin{vmatrix} B_k \\ A_k \end{vmatrix}_{r=r_k} = \begin{vmatrix} B_{k+1} \\ A_{k+1} \end{vmatrix}_{r=r_k}, \tag{7}$$

where

$$\Delta^{(k)} = \begin{vmatrix} \frac{1}{\varepsilon^k_{\theta\theta}} \frac{\partial}{\partial r} \cdot & \frac{-ik_0 \varepsilon^k_{\theta\varphi}}{\varepsilon^k_{\theta\theta}} \cdot \\ 0 & \frac{\partial}{\partial r} \cdot \end{vmatrix}, \qquad k_0 = \omega/c \text{ is the wave number;}$$

$$\text{mod} \begin{vmatrix} B \\ A \end{vmatrix}_{r\to 0} < \begin{vmatrix} M' \\ M' \end{vmatrix}; \qquad \text{mod} \begin{vmatrix} B \\ A \end{vmatrix}_{r\to\infty} \to 0. \tag{8}$$

We write (4) in the form

$$\begin{vmatrix} Y^{(k)}(r)\Psi(\theta) \\ Z^{(k)}(r)\Psi(\theta) \end{vmatrix}.$$

Then $l_r^{(k)}$ acts only on

$$\begin{vmatrix} Y \\ Z \end{vmatrix},$$

and l_θ on Ψ; therefore in them, and in (7), (8), we replace partial derivatives by total derivatives and introduce the operator L_r, generated by the differential expressions $l_r^{(k)}$, the conditions at the discontinuities

$$\Delta^{(k)} \begin{vmatrix} Y^{(k)} \\ Z^{(k)} \end{vmatrix}_{r_k} = \Delta^{(k+1)} \begin{vmatrix} Y^{(k+1)} \\ Z^{(k+1)} \end{vmatrix}_{r_k}; \qquad \begin{vmatrix} Y^{(k)} \\ Z^{(k)} \end{vmatrix}_{r_k} = \begin{vmatrix} Y^{(k+1)} \\ Z^{(k+1)} \end{vmatrix}_{r_k} \tag{9}$$

and the boundary conditions

$$\text{mod} \begin{vmatrix} Y \\ Z \end{vmatrix}_{r\to 0} < \begin{vmatrix} M'' \\ M'' \end{vmatrix}; \qquad \text{mod} \begin{vmatrix} Y \\ Z \end{vmatrix}_{r\to\infty} \to 0. \tag{9'}$$

We also introduce the operator L_θ, generated by the differential expression l_θ and the boundedness conditions at 0 and π. Then the boundary-value problem formulated above can be written as the inhomogeneous operator equation

$$L_r \begin{vmatrix} B \\ A \end{vmatrix} + L_\theta \begin{vmatrix} B \\ A \end{vmatrix} = \frac{4\pi}{c} r^2 \left| I_r \right|. \tag{10}$$

3. Method of solution.

The operator on the right-hand side of (10) is separated in the coordinates r and θ, and therefore we apply the method of normal waves [(1)], which consists in expanding spectrally

$$\begin{vmatrix} B \\ A \end{vmatrix}$$

with respect to one of the coordinates r or θ and representing it in source form with respect to the other. We shall expand into the spectrum of L_r and into source form with respect to L_θ. The resulting expansion converges considerably faster than an expansion in the spectrum of L_θ. The detour followed by Watson [(2)] was caused by the absence of a theory of non-self-adjoint operators [(3–5)]. The operator L_r is non-self-adjoint and singular. To find its spectrum one may use methods [(5)]; however, we introduce an auxiliary operator L'_r with the same $l_r^{(k)}$, but with the boundary conditions at the singular points $r = 0$ and $r = \infty$ replaced by zero conditions on spheres of small radius $r = \rho$ for $r = 0$ and of large radius

$r = R$ for $r \to \infty$. The operator L'_r is regular and acts in the space of piecewise-continuously differentiable functions. Its spectrum consists only of discrete complex eigenvalues $\{\chi_j\}$, determined by the equation

$$L'_r \begin{vmatrix} Y \\ Z \end{vmatrix} = \chi \begin{vmatrix} Y \\ Z \end{vmatrix}.$$

Except for special cases, the eigenvalues χ_j are simple. The orthogonality condition has the form:

$$\int_0^\infty Y_j(r)\, U_p(r)\, dr + \int_0^\infty Z_j(r)\, V_p(r)\, dr = \begin{cases} N_j, & j = p, \\ 0, & j \neq p, \end{cases} \tag{11}$$

where

$$\begin{vmatrix} U_p \\ V_p \end{vmatrix}$$

are the eigenfunctions of the adjoint operator L'^*_r, determined from Lagrange' s identity (4). Since the eigenvalues μ_j of the operator L'^*_r are equal to $\bar{\chi}_j$ (the bar denotes complex conjugation), it is not difficult to show that (11) reduces to the form

$$\int_0^\infty \frac{Y_j Y_p}{\varepsilon_{rr} r^2}\, dr + \int_0^\infty \frac{Z_j Z_p}{r^2}\, dr = \begin{cases} N_j, & j = p, \\ 0, & j \neq p; \end{cases} \tag{11'}$$

the quantity N_j will be called the normalizing factor.

By virtue of the theorem of M. V. Keldysh (3), we expand the solution (10) in the form

$$\begin{vmatrix} B \\ A \end{vmatrix} = \sum_{j=0}^{\infty} \begin{vmatrix} Y_j(r) \\ Z_j(r) \end{vmatrix} \Phi_j(\theta). \tag{12}$$

For the Fourier coefficients of this expansion, by virtue of (10), (11), we obtain the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Phi_j}{d\theta}\right) + \chi_j\Phi_j = \frac{4\pi}{cN_j}\int_0^{\infty}\frac{I_r Y_j}{\varepsilon_{rr}}\,dr. \tag{13}$$

Representing its solution, in the usual way, with the aid of Green' s function, we obtain the general solution in the form

$$\begin{vmatrix} B \\ A \end{vmatrix} = -\frac{2\pi^2}{c}\sum_{j=0}^{\infty}\begin{vmatrix} Y_j \\ Z_j \end{vmatrix}\frac{1}{\sin(\nu_j\pi)N_j}\left\{P_{\nu_j}[\cos(\pi-\theta)]\int_0^{\theta} I_j P_{\nu_j}(\cos\theta')\sin\theta'\,d\theta' + \right.$$

$$\left. + P_{\nu_j}[\cos\theta]\int_0^{\theta} I_j P_{\nu_j}\cos(\pi-\theta')\sin\theta'\,d\theta'\right\}, \tag{14}$$

where $\chi_j = \nu_j(\nu_j+1)$, and for the case when the field is excited by a Hertz dipole at the point $\theta = 0,\ r = b$:

$$\begin{vmatrix} B \\ A \end{vmatrix} = -\frac{\pi P}{cb^2}\sum_{j=0}^{\infty}\begin{vmatrix} Y_j(r) \\ Z_j(r) \end{vmatrix}\frac{Y_j(b)}{N_j\sin\nu_j\pi}P_{\nu_j}[\cos(\pi-\theta)], \tag{14'}$$

where P is the electric moment of the Hertz dipole. Expanding P_{ν_j} into the sum

$$\frac{1}{\pi i}\left\{L_{\nu_j}^{(1)}[\cos(\pi-\theta)] - L_{\nu_j}^{(2)}[\cos(\pi-\theta)]\right\}$$

and using the asymptotic representation

$$L_{\nu_j}^{(1,2)} = Q_{\nu_j} \pm i\frac{\pi}{2}P_{\nu_j} \sim \sqrt{\frac{\pi}{2\nu_j\sin\theta}}\,e^{\pm i[\nu_j^*\theta+\pi/4]}, \qquad \nu^* = \nu + \frac{1}{2},$$

we obtain for $(14')$ the approximate expression

$$\begin{vmatrix} B \\ A \end{vmatrix} = \frac{2P}{cb^2}\sqrt{\frac{\pi}{2\sin\theta}}\sum_{j=0}^{\infty}\begin{vmatrix}\dot{Y}_j(r)\\ Z_j(r)\end{vmatrix}\frac{Y_j(b)\nu_j^{1/2}}{N_j}\left\{\sum_{n=0}^{\infty} e^{i(n+1)2\pi\nu_j - i(\nu_j^*\theta+\pi/4)} + e^{i2\pi n\nu_j + i(\nu_j^*\theta+\pi/4)}\right\}, \tag{15}$$

where $1/\sin\nu_j\pi$ has been expanded in a series—a geometric progression.

Each term of (15) is a normal wave running along the layers $\|\varepsilon\| = \text{const}$, characterized by the wave number α_j and the attenuation coefficient β_j, which are determined by the complex eigenvalue χ_j according to the formula $\nu_j^* = \alpha_j + i\beta_j = \sqrt{\chi_j + 1/4}$, and also by the distribution of amplitudes over the wave front,

$$\begin{vmatrix} Y_j(r) \\ Z_j(r) \end{vmatrix};$$

n indicates the number of circuits of the wave around the sphere; waves with $n \neq 0$ are called round-the-world echoes. Because of attenuation, the waves with $n = 0$, arriving at the observation point θ along the shortest arc, stand out. For these waves the electromagnetic field is expressed by Dirichlet series:

$$\begin{aligned} H_\varphi &= Ai\sum_{j=0}^{\infty} Y_j(r)B_j; \qquad E_r = -\frac{Ai}{k_0 r}\sum_{j=0}^{\infty}\nu_j Y_j(r)B_j; \qquad E_\theta = \frac{A}{k_0}\sum_{j=0}^{\infty}\frac{dY_j}{dr}B_j; \\ E_\varphi &= Ai\sum_{j=0}^{\infty} Z_j(r)B_j; \qquad H_r = \frac{Ai}{k_0 r}\sum_{j=0}^{\infty}\nu_j Z_j(r)B_j; \qquad H_\theta = -\frac{A}{k_0}\sum_{j=0}^{\infty}\frac{dZ_j}{dr}B_j, \end{aligned} \tag{16}$$

where

$$A = \frac{2P}{cb^2 r}\sqrt{\frac{\pi}{2\sin\theta}}\, e^{-i\omega t}; \qquad B_j = \frac{Y_j(b)}{N_j}\nu_j^{1/2} e^{i(\nu_j^*\theta+\pi/4)}.$$

4. Of greatest interest is the case of such a distribution of $\|\varepsilon\|$ in r in which intervals (r_i, r_{i+1}) with a slow variation of $\|\varepsilon\|$ alternate with intervals (r_{i+1}, r_{i+2}), $i = 0, 2, 4, 6, \ldots$, where $\|\varepsilon\|$ changes sharply over the length of one wavelength, which leads to a noticeable reflection of waves at the boundaries of the intervals (r_i, r_{i+1}). In this case the spectrum χ_j of the operator L'_r is divided into a number of branches, each of which is associated with the corresponding interval (r_i, r_{i+1}). The forms of the normal waves $|Y_j/Z_j|^{(i)}$ of the branch $\chi_j^{(i)}$ are localized in the interval (r_i, r_{i+1}). A dipole placed in the interval (r_i, r_{i+1}) will produce a field that will propagate along θ, remaining localized within (r_i, r_{i+1}). Such propagation is characteristic of waveguide channels ([1]). Only the first types of waves $\chi_j^{(i)}$ have small attenuation; waves of higher numbers $j > N^{(i)}$, for which the period of spatial modulation of $|Y_j/Z_j|$ along the front

$\theta = \text{const}$ is less than the wavelength in the given waveguide (r_i, r_{i+1}), have larger $\beta_j^{(i)}$, and the series $(14')$ for approximate calculations of the field may be cut off at $N^{(i)}$. The filtering of high types of waves in waveguide channels ensures the rapid convergence of the series $(14')$.

As $\rho \to 0$ and $R \to \infty$, the discrete $\chi_j^{(i)}$ of the branches $i = 1, 2, \dots$ of the operator L'_ρ continuously pass into the corresponding $\chi_j^{(i)}$ of the operator L_r. Only on one of the branches, $i = 0$, associated with the singularity at the point $r = 0$, do the $\chi_j^{(0)}$ come closer together, turning in the limit into a continuous spectrum; and the corresponding sum $(14')$ becomes an integral with respect to $d\chi$, which describes waves penetrating the central part of the medium. If the latter has losses, then such waves may be neglected.

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Note: Figure translations are in progress. See original paper for figures.