V. M. POLYAKOVA

Let us consider the equation:

1961

SovietRxiv

Abstract

Full Text

V. M. POLYAKOVA

ON LEVEL LINES OF A SOLUTION OF AN ELLIPTIC EQUATION

(Presented by Academician I. G. Petrovskii on 25 V 1961)

Let us consider the equation:

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\,\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}$$
$$+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=0, \tag{1}$$

defined in the strip $K_h=\{0\le x\le\infty,\ -h\le y\le h\}$, $0<h<1$. Let A,B,C be twice continuously differentiable, D and E continuously differentiable, and, together with the indicated derivatives, bounded in absolute value by one; moreover, $-1<F\le 0$. Suppose that the equation is uniformly elliptic:

$$A\xi^2+2B\xi\eta+C\eta^2\ge\alpha(\xi^2+\eta^2),\qquad \alpha>0, \tag{2}$$

for arbitrary ξ and η.

Let $u(x,y)$ be a solution of equation (1), and let $|u(x,y)|<1$ in the strip K_h. Denote by Ω the level set $u(x,y)=0$. We shall call a component L of the set Ω **proper** if it starts on the left side of K_h and goes to infinity without touching its upper or lower sides. Let L_1 and L_2 be proper components of Ω, with L_1 lying above L_2. Denote by $y_1(x_0)$ and $y_2(x_0)$, respectively, the upper point of intersection of L_1 and the lower point of intersection of L_2 with the line $x=x_0$. Put $\rho(x_0)=y_1(x_0)-y_2(x_0)$.

Theorem. Let L_1,L_2 be proper components. Then

$$\lim_{x\to\infty}\left[\ln\frac{1}{\rho(x)}:x\right]<\infty. \tag{3}$$

Proof. Suppose that (3) is false. This means that there exists a function $g(x)\to\infty$ as $x\to\infty$ such that

$$\rho(x)\le e^{-xg(x)}. \tag{3a}$$

Denote by G the closed region lying between L_1 and L_2. We may assume that inequality (3a) holds starting from $x = 0$, and that $g(x)$ increases monotonically. In the proof we shall use two lemmas analogous to Lemmas 1.6.1 and 1.1.1 from (1).

Let h, $0 < h < 1$, be an arbitrary number. Let Γ_h be a Γ-shaped strip: $x_0 - h < x < +\infty$ for $y_0 - h < y < y_0 + h$; $x_0 - h < x < x_0 + h$ for $-\infty < y < y_0 + h$. Suppose that in the strip Γ_h there are two smooth curves Γ_1 and Γ_2, each of which has limit points on both sides of Γ_h. Suppose they do not intersect, and that Γ_1 is situated above or to the right of Γ_2; denote by $G^{(1)}$ the region lying between them, and by $G^{(2)}$ the part of the strip Γ_h not separated from infinity by the curve Γ_2.

Lemma 1. Let $N > 2^{10}h$ be an arbitrary number and suppose that at least one of the following assumptions holds:

1) In the domain $G^{(1)} + G^{(2)}$ there is defined a solution $u(x, y)$ of equation (1) such that: a) $|u(x, y)||_{\Gamma_1} > a > 0$; b) $|u(x, y)||_{G^{(2)}} > a \cdot 2^{-2N/h}$; c) $|u(x, y)| < 1$ in $G^{(1)}$. We assume that Γ_2 is projected one-to-one onto the y-axis.
2) In the domain $G^{(1)}$ there is situated a subdomain g, also having boundary points on both sides of Γ_h; γ is the part of its boundary lying strictly inside Γ_h. In the domain $G^{(1)}$ there is defined a solution of equation (1), $u(x, y)$, such that: a) $|u(x, y)||_{\Gamma_1} > a > 0$; b) $u(x, y)|_\gamma = 0$, $|u(x, y)| < a \cdot 2^{-2N/h}$ for $(x, y) \in g$; c) $|u(x, y)| < 1$ in $G^{(1)}$.

Then

$$\mu^2 G^{(1)} > Nh/M_1, \tag{4}$$

M_1 depends only on α of inequality (2).

The proof of this lemma is obtained from the proof of Lemma 1.6.1 of [1] by replacing the strip $P_h = \{-h < y < h\}$ by the Γ-shaped strip Γ_h.

Lemma 2. Let a domain G be situated in the disk Q_R of radius $R < 1$, containing the center O of the disk and having boundary points on the boundary of the disk. Let Γ be the part of the boundary of the domain G which lies strictly inside Q_R. Let equation (1) be defined in G.

There exists a constant M_2, depending only on the constant α of inequality (2), such that if $\mu^2 G < R^2/M_2$, then for any positive solution $u(x, y)$ of equation (1), continuous in G and vanishing on the boundary Γ, the inequality

$$u(0) < \frac{1}{2} \max_{P \in \bar{G}} u(P) \tag{5}$$

holds.

This is Lemma (1.1.1) from [1].

Proof of the theorem.

1°. Denote:

$$\max_{y\in G} u(0,y) = a.$$

Suppose that, starting with $x = x_0$, the inequality

$$\frac{x}{2}\, g\Big(\frac{x}{2}\Big) > 1$$

holds.

Put

$$X = \max(4M_2, x_0).$$

We shall prove that for every $x > X$ the inequality

$$\max_{y\in G} u(x,y) < a \cdot 2^{\frac{-\exp\left(\frac{x}{2}\, g\left(\frac{x}{2}\right)\right)}{2}}. \tag{6}$$

First we shall show that

$$\max_{y\in G} u(x_1,y) > \max_{y\in G} u(x_2,y), \quad \text{if } x_1 < x_2. \tag{7}$$

Fix some number R, $0 < R < 1$. Since $\rho(x)$ decreases as x increases, there will be an x' such that for every $x'' > x'$ the area of the domain G situated in the disk Q_R of radius R with center at (x'', y), $y \in G$, will be less than R^2/M_2. Then by Lemma 2

$$\max_{(x,y)\in\bar{G}\cap\bar{Q}_R} u(x,y) > 2\max_{y\in G} u(x'',y).$$

By the maximum principle, $\max\limits_{(x,y)\in\bar{G}\cap\bar{Q}_R} u(x,y)$ is attained at a point (x_1, y_1) on the boundary of the domain, and moreover on that part of it which lies on the boundary of Q_R. Hence either $x_1 > x''$ or $x_1 < x''$. Suppose $x_1 > x''$. Taking the point (x_1, y_1) as the new center of a disk of radius R, we obtain that the maximum of $u(x,y)$ in the intersection

the domain G with the newly obtained circle, by the maximum principle, must also lie to the right of $x = x_1$ and, by Lemma 2, be at least twice as large. Hence the function must increase without bound as $x \to \infty$. Consequently, $x_1 < x''$, and (7) follows from the maximum principle.

Let us now prove the required inequality (6). Take on $[0, x]$, $x > X$, the following points:

$$x^{(1)} = 2M_2 e^{-g \cdot 0},$$

$$\dots\dots\dots\dots\dots\dots$$

$$x^{(n)} = x^{(n-1)} + 2M_2 e^{-x^{(n-1)} g\left(x^{(n-1)}\right)}.$$

Then their number n satisfies the inequality

$$n > \left[\frac{x}{2 \cdot 2M_2 e^{-\frac{x}{2} g\left(\frac{x}{2}\right)}}\right] > \left[e^{\frac{x}{2} g\left(\frac{x}{2}\right)}\right] > \frac{1}{2} e^{\frac{x}{2} g\left(\frac{x}{2}\right)}, \tag{8}$$

since $x > X$.

On the other hand,

$$\max_{y \in G} u\left(x^{(k)}, y\right) > 2 \max_{y \in G} u\left(x^{(k+1)}, y\right).$$

Indeed, for each $x^{(k)}$ we find a point $P^{(k)}$ at which the maximum of $u(x^{(k)}, y)$, $y \in G$, is attained. Construct circles $Q^{(k)}$ with radii

$$r_k = 2M_2 e^{-x^{(k-1)} g\left(x^{(k-1)}\right)}$$

and centers at $P^{(k)}$. Then

$$\mu_2 Q^{(k)} \cap G < 4e^{-2x^{(k-1)} g\left(x^{(k-1)}\right)} M_2 = \frac{r_k^2}{M_2};$$

hence Lemma 2 is applicable to each $Q^{(k)}$. Therefore,

$$\max_{\overline{Q}^{(k)} \cap \overline{G}} u(x, y) > 2u\left(P^{(k)}\right).$$

Hence, all the more,

$$u\left(P^{(k-1)}\right) > 2u\left(P^{(k)}\right),$$

but since n satisfies condition (8), it follows that

$$\max u(x,y) < a \cdot 2^{-\frac{\exp\left[\frac{x}{2}\, g\left(\frac{x}{2}\right)\right]}{2}},$$

which was required to be proved.

2°. Let $(0, y_0)$ be the point at which the maximum of $u(0,y)$ for $y \in G$ is attained; $u(0,y_0) = a_0$. By the continuity of $u(x,y)$ there exists a number $\delta > 0$ such that

$$\min_{y_0-\delta<y<y_0+\delta} u(0,y) > \frac{a_0}{2}.$$

Denote $h_1 = h - |y_0| - \delta$. Since $u(x,y) = 0$ on L_1 and L_2 (the boundary of the domain G), we have $h_1 > 0$. We show that for all $x > X + 2h_1$ in the strip $K_\delta = \{y_0 - \delta < y < y_0 + \delta\}$ the inequality

$$|u(x,y)| \le a_1(x)\, 2^{2N/h_1} = a_2, \tag{9}$$

holds, where

$$a_1(x) = a_0 \cdot 2^{-\frac{\exp\{(x/2-h_1)g(x/2-h_1)\}}{2}}, \qquad N = \max(2^{10}h_1,\, 8M_1h).$$

Suppose that this is not so. Let there exist a point $(\hat{x}, \hat{y}) \in K_\delta$, $\hat{x} > X+2h_1$, such that $|u(\hat{x},\hat{y})| > a_1(\hat{x})2^{2N/h_1}$. Let $u(\hat{x},\hat{y}) > 0$ (the case $u(x,y) < 0$ is symmetric). By the Kronrod-Landis theorem (2), for almost all c from the range of values of the function $u(x,y)$, the level line $u(x,y) = c$ contains no points with zero gradient of the function $u(x,y)$. Therefore one can find a point $(\hat{x},\hat{y})$ such that $u(\hat{x},\hat{y}) > a_2$ and at the same time, on the level line of the function $u(x,y)$ passing through this point, $\operatorname{grad} u(x,y) \ne 0$. Then the component l of the indicated level line passing through the point $(\hat{x},\hat{y})$ behaves as follows: a) either it intersects the left side of the strip $\hat{x} - 2h_1 < x < \hat{x}$ (the right side of the strip $\hat{x} < x < \hat{x} + 2h_1$), b) or it exits onto the boundary of the strip K_h. In any of these cas-

than it separates the Γ-shaped strip $\Gamma_{h_1} = \{\hat{x} - 2h_1 < x < \hat{x}$ for $y < \tilde{y};\ x > \hat{x} - 2h_1$ for $\tilde{y} + 2h_1 > y > \tilde{y}\}$, or the strip $\Gamma_{h_1} = \{-\infty < x < \hat{x} + 2h_1$ for $\tilde{y} - 2h_1 < y < \tilde{y};\ \hat{x} < x < \hat{x} + 2h_1$ for $y > \tilde{y}\}$. From 1° it follows that, for all $x > X$, the inequality

$$\max_{y\in G} u(x,y) < a_1(x)$$

holds. Hence Lemma 1 (case 2) is applicable to the strip Γ_{h_1}; here the part of G in the strip Γ_{h_1} plays the role of g. Consequently, the area σ between the component under consideration of the level line l and the part of G in the strip Γ_{h_1} satisfies inequality (4), $\sigma > Nh_1/M_1$. On the other hand, $\sigma < 8hh_1$. But, recalling that $N = \max(2^{10}h_1, 8M_1h)$, we arrive at a contradiction. This proves the required inequality (9).

Let us bring (9) to a somewhat different form:

$$a_1 2^{2N/h_1} = a_0 \cdot 2^{-\frac{\exp[(x/2-h_1)g(x/2-h_1)]}{2}} \cdot 2^{2N/h_1} < a_0 \cdot 2^{-2^{\frac{x}{4}g\left(\frac{x}{4}\right)}},$$

starting from some X_1. Hence,

$$u(x,y)_{\,y_0-\delta<y<y_0+\delta} < a_0 \cdot 2^{-2^{\frac{x}{4}g\left(\frac{x}{4}\right)}}, \qquad x > \max(X, X_1). \tag{10}$$

3°. Let us prove the theorem. We have:

$$u(0,y)_{\,y_0-\delta<y<y_0+\delta} > \frac{a_0}{2} = a',$$

$$|u(x_0,y)|_{\,y_0-\delta<y<y_0+\delta} < a_0 \cdot 2^{-2^{\frac{x}{4}g\left(\frac{x}{4}\right)}} < a' \cdot 2^{-2^{\frac{x_0}{8}g\left(\frac{x_0}{4}\right)}}, \qquad x_0 > \max(X_1, X).$$

Then, by Lemma 1, the area S between the straight lines $x = 0$ and $x = x_0$, $x_0 > \max(X, X_1)$, in the strip K_δ must be greater than

$$\frac{x_0}{8}\, g\!\left(\frac{x_0}{4}\right)\frac{\delta^2}{M_1}.$$

But $S < 2\delta x_0$, which is impossible for x_0 satisfying the inequality $g(x_0/4) > 16M_1/\delta$. We arrive at a contradiction, which proves the theorem.

Corollary. Let the equation $\Delta u = 0$ be given. Let $|u| < 1$ in an angle φ (or in a part of an angle $\varphi : r > r_0$). The greatest possible rate of decrease of the distance between a pair of lines of one and the same level set $u = \text{const}$, lying entirely in the angle φ, is

$$\rho(r) = \frac{M}{r^\alpha}, \qquad \alpha > 0.$$

Proof. It is clear that for the Laplace equation the theorem is valid without the restriction $h < 1$, since the equation is invariant under similarity transformations, and also, instead of the zero level set, one may consider the level set $u(x,y) = \text{const}$. Make the transformation:

$$x_1 = \operatorname{Re}\ln(x+iy), \qquad y_1 = \operatorname{Im}\ln(x+iy).$$

The equation $\Delta u = 0$ is invariant under it, i.e. $\Delta u(x_1, y_1) = 0$. Then the region inside the angle φ passes into a half-strip of width φ, and

$$\rho(x, y) = \tilde{\rho}(x_1, y_1) = Me^{-\alpha u}.$$

But this is the greatest possible rate of decrease of the distance between level lines $u(x, y) = \text{const}$ lying entirely in the strip $(+\infty > x > x_0,\ y_2 < y < y_1)$.

Received
14 III 1961

REFERENCES

1. E. M. Landis, *UMN*, **14**, no. 1 (85) (1959).

2. A. S. Kronrod, E. M. Landis, *DAN*, **58**, No. 7 (1947).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.