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Abstract

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ON THE BEHAVIOR OF THE SOLUTION OF THE STEFAN PROBLEM UNDER UNBOUNDED INCREASE OF TIME

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In the present note the behavior of the solution of the Stefan problem with a Cauchy condition for a parabolic equation as $t \to \infty$ is studied. For the heat-conduction equation, the asymptotic behavior as $t \to \infty$ of the solution of the Stefan problem with boundary conditions was considered in ([1]).

We formulate the Stefan problem with a Cauchy condition as follows: find functions $s(t)$ and $u_i(x,t)$ such that $u_i(x,t)$ $(i=1,2)$ are bounded and continuous in the domains $\overline{D_i}$ ($D_1 = \{x < s(t),\, t > 0\}$ and $D_2 = \{x > s(t),\, t > 0\}$) everywhere, except possibly at the point $(0,0)$, have derivatives $\partial u_i/\partial x$, $\partial^2 u_i/\partial x^2$, $\partial u_i/\partial t$ in D_i, and satisfy the equation

$$L_i(u_i) \equiv \frac{\partial^2 u_i}{\partial x^2} + B_i(x,t)\frac{\partial u_i}{\partial x} + C_i(x,t)u_i - A_i(x,t)\frac{\partial u_i}{\partial t} = 0 \quad \text{in } D_i \qquad (1)$$

and the conditions

$$u_1(x,0) = \varphi_1(x) \leqslant 0 \quad \text{for } x < 0; \qquad u_2(x,0) = \varphi_2(x) \geqslant 0 \quad \text{for } x > 0; \qquad (2)$$

$$u_i(s(t),t) = 0; \qquad \left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right)_{x=s(t)} = \frac{ds(t)}{dt}, \qquad (3)$$

where $A_i(x,t) \geqslant a > 0$; $C_i(x,t) \leqslant 0$; A_i, B_i, C_i, φ_i are sufficiently smooth and bounded functions; $s(t)$ is a differentiable function, $s(0) = 0$.

Using the maximum principle, it is easy to prove the lemma (see ([1])):

Lemma. Let $y(t)$ be a differentiable function, and let $V_i(x,t)$ $(i=1,2)$ be bounded and continuous in the domains $\overline{\Omega_i}$ ($\Omega_1 = \{x < y(t),\, t > 0\}$, $\Omega_2 = \{x > y(t),\, t > 0\}$) everywhere except possibly at the point $(y(0),0)$, and suppose that $V_i(x,t)$ have derivatives $\partial V_i/\partial x$, $\partial^2 V_i/\partial x^2$, $\partial V_i/\partial t$ in Ω_i. Let, further, $u_i(x,t)$, $s(t)$ be a solution of problem (1)–(3), and let $L_i(V_i) \leqslant 0$ in Ω_i $(i=1,2)$,

$$u_1(x,0) \leqslant V_1(x,0) \quad \text{for } x < y(0); \qquad u_2(x,0) \leqslant V_2(x,0) \quad \text{for } x > s(0);$$

$$V_i(y(t),t) = 0; \qquad \frac{dy(t)}{dt} < \left(\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x}\right)_{x=y(t)} \quad \text{for } 0 \leqslant t \leqslant T,$$

and let there exist $t_0 > 0$ such that $y(t) \leqslant s(t)$ for $0 < t \leqslant t_0$. Then for all $t \in (0,T)$ the inequality $y(t) \leqslant s(t)$ holds.

It is clear that if one changes the sign of the inequalities in all conditions of the lemma, then one obtains the relation $y(t) \geqslant s(t)$ for $t \in (0,T)$.

Using the similarity method (see ([2])), one can prove the following proposition:

Theorem 1. The solution of the equation $\partial^2 u_i/\partial x^2 = a_i^2 \partial u_i/\partial t$ with the conditions $u_1(x,0) = u_- < 0$ for $x < 0$, $u_2(x,0) = u_+ > 0$ for $x > 0$ and the conditions (3) has the form $s(t) = \alpha_0\sqrt{t}$ and

$$u_1(x,t) = u_-\,[\mathrm{erf}(a_1\alpha_0/2) - \mathrm{erf}(a_1 x/2\sqrt{t})]/[1 + \mathrm{erf}(a_1\alpha_0/2)] \quad \text{for } x < s(t),$$

$$u_2(x,t) = u_+\,[\mathrm{erf}(a_2 x/2\sqrt{t}) - \mathrm{erf}(a_2\alpha_0/2)]/[1 - \mathrm{erf}(a_2\alpha_0/2)] \quad \text{for } x > s(t),$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi;$$

α_0 is the root of the equation

$$F(\alpha; u_-, u_+; a_1, a_2) \equiv \frac{a_1 u_- e^{-a_1^2\alpha^2/4}}{1 + \mathrm{erf}(a_1\alpha/2)} + \frac{a_2 u_+ e^{-a_2^2\alpha^2/4}}{1 - \mathrm{erf}(a_2\alpha/2)} + \frac{\alpha\sqrt{\pi}}{2} = 0.$$

It is easy to see that $\dot{F}_\alpha > 0$ and $\alpha_0 < 0$ when $a_1 u_- + a_2 u_+ > 0$; $\alpha_0 > 0$ when $a_1 u_- + a_2 u_+ < 0$.

Theorem 2. Let $u_i(x,t)$ be a solution of the equation

$$L_i(u_i) \equiv \partial^2 u_i/\partial x^2 + B_i(x,t)\partial u_i/\partial x - A_i(x,t)\partial u_i/\partial t = 0$$

with conditions (2) and (3), and let the coefficients $A_i(x,t) \to a_i^2$, $B_i(x,t) = o(1/\sqrt{t})$ as $t \to \infty$, uniformly in x. Suppose $xB_i(x,t) \geq 0$ for $|x| \geq x_0$, where

$x_0 > 0$ is a certain constant. Then, if $\varphi_1(x) \to u_- < 0$ as $x \to -\infty$ and $\varphi_2(x) \to u_+ > 0$ as $x \to \infty$, with $a_1u_- + a_2u_+ > 0$, then $\lim_{t\to\infty} s(t) = -\infty$ and

$$\lim_{t\to\infty} u_2(x,t) = u_+ \operatorname{erf}(a_2|\alpha_0|/2)/[1+\operatorname{erf}(a_2|\alpha_0|/2)] \equiv U_+.$$

Proof. Since $a_1u_- + a_2u_+ > 0$, we have $\alpha_0 < 0$ and $F(\alpha_1; u_-, u_+; a_1, a_2) < 0$, where $\alpha_1 = \alpha_0 - \varepsilon$ and $\varepsilon > 0$. Construct the functions $y(t) = \alpha_1\sqrt{t} - N$ and

$$V_1(x,t) = (u_- + 3\varepsilon_1)\frac{\operatorname{erf}(e_1\alpha_1/2) - \operatorname{erf}(e_1(x+N)/2\sqrt{t})}{1+\operatorname{erf}(e_1\alpha_1/2)} + \varepsilon_2 v(x,t)$$

for $x < y(t)$,

$$V_2(x,t) = (u_+ + 3\varepsilon_1)\frac{\operatorname{erf}(e_2(x+N)/2\sqrt{t}) - \operatorname{erf}(e_2\alpha_1/2)}{1-\operatorname{erf}(e_2\alpha_1/2)} + \varepsilon_2 v(x,t)$$

for $x > y(t)$, where $e_i^2 = a_i^2 - \varepsilon_3$ and $v(x,t) = e^{-\theta(x+N)^2/t} - e^{-\theta\alpha_1^2}$. Choose θ so small that

$$L_i(v) \le -[\gamma + 2\theta(x+N)B_i]\exp(-\theta(x+N)^2/t)/t,$$

where $\gamma > 0$ is a constant and $\theta < \min(a_1^2/8, a_2^2/8)$. In this case

$$\left(\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x}\right)_{x=y(t)} = \frac{-1}{\sqrt{\pi t}}\left[\frac{e_1(u_- + 3\varepsilon_1)e^{-e_1^2\alpha_1^2/4}}{1+\operatorname{erf}(e_1\alpha_1/2)} + \frac{e_2(u_+ + 3\varepsilon_1)e^{-e_1^2\alpha_1^2/4}}{1-\operatorname{erf}(e_2\alpha_1/2)}\right],$$

$$L_1(V_1) =$$

$$= \frac{-e_1(u_- + 3\varepsilon_1)}{\sqrt{\pi}[1+\operatorname{erf}(e_1\alpha_1/2)]}\left[(A_1(x,t) - e_1^2)\frac{x+N}{2t\sqrt{t}} + \frac{B_1(x,t)}{\sqrt{t}}\right]e^{-e_1^2(x+N)^2/4t} + \varepsilon_2 L_1(v),$$

$$L_2(V_2) =$$

$$= \frac{e_2(u_+ + 3\varepsilon_1)}{\sqrt{\pi}[1-\operatorname{erf}(e_2\alpha_1/2)]}\left[(A_2(x,t) - e_2^2)\frac{x+N}{2t\sqrt{t}} + \frac{B_2(x,t)}{\sqrt{t}}\right]e^{-e_2^2(x+N)^2/4t} + \varepsilon_2 L_2(v).$$

First choose $\varepsilon_1 > 0$ and $\varepsilon_3 > 0$ such that $u_- + 3\varepsilon_1 < 0$, $F(\alpha_1; u_- + 3\varepsilon_1, u_+ + 3\varepsilon_1; e_1, e_2) < 0$, and $\varepsilon_3 < \min(a_1^2/2, a_2^2/2)$. Then choose $\varepsilon_2 > 0$ sufficiently small and $k > 0$ large so that $2\varepsilon_1 - \varepsilon_2 e^{-\theta\alpha_1^2} > 0$ and

$$(u_+ + 3\varepsilon_1)\,\frac{\operatorname{erf}(e_2 k/2) - \operatorname{erf}(e_2\alpha_1/2)}{1 - \operatorname{erf}(e_2\alpha_1/2)} + \varepsilon_2(e^{-\theta k^2} - e^{-\theta\alpha_1^2}) \geq u_+ + \varepsilon_1.$$

For such $\varepsilon, \varepsilon_1, \varepsilon_2, \theta$, and k, decrease ε_3 so that

$$\frac{e_2(u_+ + 3\varepsilon_1)(1+k)\varepsilon_3}{\sqrt{\pi}[1 - \operatorname{erf}(e_2\alpha_1/2)]} - \frac{e_1(u_- + 3\varepsilon_1)\varepsilon_3}{\sqrt{\pi}[1 + \operatorname{erf}(e_1\alpha_1/2)]} + 2\varepsilon_2\theta k\varepsilon_3 \leq \varepsilon_2\gamma.$$

Moreover, take all ε_i so that $\varepsilon_i \to 0$ as $\varepsilon \to 0$. For any ⋯

for arbitrary ε_1 and ε_3, in view of the condition $xB_i(x,t) \geqslant 0$ for $|x| \geqslant x_0$, as in (3), one can find such a $T > 0$ that $u_2(x,t) \leqslant u_+ + \varepsilon_1$, $|A_i(x,t) - a_i^2| \leqslant \varepsilon_3$, and $|B_i(x,t)| \leqslant \varepsilon_3/\sqrt{t}$ for $t \geqslant T$. For such a T we have: $u_1(x,T) \leqslant u_- + \varepsilon_1$ for $x \leqslant -N$, where $N > 0$ is sufficiently large and such that $-N < s(T)$ and $N > x_0$ (see (4)). For simplicity we assume that $T = 0$. Then it is easy to verify that $V_1(x,0) \geqslant u_1(x,0)$, $V_2(x,t)|_{x=k\sqrt{t}-N} \geqslant u_2(x,t)|_{x=k\sqrt{t}-N}$, $V_i(y(t),t) = 0$, $(\partial V_1/\partial x - \partial V_2/\partial x)_{x=y(t)} > dy(t)/dt$, and $L_1(V_1) \leqslant 0$ for $x < y(t)$, $L_2(V_2) \leqslant 0$ for $y(t) < x \leqslant k\sqrt{t} - N$. Hence it is easy to see that $s(t) \geqslant y(t)$. Consequently,

$$\lim_{t\to\infty} u_2(x,t) \leqslant \lim_{t\to\infty} V_2(x,t) = (u_+ + 3\varepsilon_1)$$

$$\times \operatorname{erf}(e_2|\alpha_1|/2)/[1 + \operatorname{erf}(e_2|\alpha_1|/2)] + \varepsilon_2(1 - e^{-\theta a_1^2}).$$

Since ε is arbitrary and $\varepsilon_i \to 0$ as $\varepsilon \to 0$, it follows that $\lim\limits_{t\to\infty} u_2(x,t) \leqslant U_+$. Analogously one can prove that $\lim\limits_{t\to\infty} u_2(x,t) \geqslant U_+$ and $\lim\limits_{t\to\infty} s(t) = -\infty$.

If $B_i(x,t) = O(1/\sqrt{t})$, then Theorem 2 may turn out to be false. Indeed, take $A_i \equiv 1$, $B_1 \equiv 0$, $B_2 = f(x/2\sqrt{t})/\sqrt{t}$, $\varphi_1 \equiv 0$ and $\varphi_2 \equiv 1$, where $f(z) = 0$ for $z \geqslant 1$; $f(z) = 2z - 2$ for $0 \leqslant z \leqslant 1$; $f(z) = -2$ for $z \leqslant 0$. Then

$$s(t) = \alpha_1\sqrt{t}, \quad u_1(x,t) = 0, \quad u_2(x,t) = [\Phi(x/2\sqrt{t}) -$$

$$-\Phi(\alpha_1/2)]/[\Phi(\infty) - \Phi(\alpha_1/2)], \quad \text{where} \quad \Phi(x) = \int_0^x \exp(-z^2 - G(z))\,dz, \quad G(z) =$$

$$= -2\int_0^x f(x)\,dx, \quad \alpha_1 \text{ is the root of the equation} \quad \exp(-\alpha^2/4 - G(\alpha/2))/[\Phi(\infty) -$$

$$-\Phi(\alpha/2)] + \alpha = 0.$$

It is easy to verify that $\lim\limits_{t\to\infty} u_2(x,t) \neq U_+$.

Theorem 3. Let $u_i(x,t)$ be the solution of problem (1)–(3), and let the coefficients $A_i(x,t) \to a_i^2$, $B_i(x,t) = o(1/\sqrt{t})$ as $t \to \infty$ uniformly in x, with

$xB_1(x,t) \geqslant 0$ for $|x| \geqslant x_0$, $C_1(x,t) \equiv 0$, $C_2(x,t) \leqslant C < 0$. Then, if $\varphi_1(x) \to u_- < 0$ as $x \to -\infty$ and $\varphi_2(x)$ is bounded, then

$$\lim_{t\to\infty} s(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} u_1(x,t) = u_-\ \mathrm{erf}(a_1\alpha_1/2)/[1+\mathrm{erf}(a_1\alpha_1/2)] \equiv U_-,$$

where α_1 is the root of the equation $F(\alpha; u_-, 0; a_1, a_2) = 0$.

It remains for us to prove that $u_2(x,t) \leqslant \varepsilon_1$ for $x > s(t)$ and $t \geqslant T$, since the rest of the proof can be carried out in the same way as in Theorem 2. For this purpose construct the functions $w_\pm(x,t) = Me^{-\theta t} \pm u_2(x,t)$. Then $L_2(w_\pm) = M(A_2(x,t)\theta + C_2(x,t))e^{-\theta t} \leqslant 0$ for $x > s(t)$, $w_\pm(x,0) = M \pm \varphi_2(x) \geqslant 0$, and $w_\pm(x,t)|_{x=s(t)} \geqslant 0$ for sufficiently small θ and large M. By the maximum principle it follows that $|u_2(x,t)| \leqslant Me^{-\theta t}$, whence $u_2(x,t) \leqslant \varepsilon_1$ for $x > s(t)$ and $t \geqslant T$.

Theorem 3 is also valid in the case when $A_2(x,t)$ and $B_2(x,t)$ are only bounded.

Theorem 4. Let $u_i(x,t)$ be the solution of problem (1)–(3), and let the coefficients $A_i(x,t) \to a_i^2$, $B_i(x,t) = o(1/\sqrt{t})$ as $t \to \infty$ uniformly in x, with $B_2(x,t) \leqslant -b < 0$, $xB_1(x,t) \geqslant 0$ for $|x| \geqslant x_0$, $C_1(x,t) \equiv 0$, $C_2(x,t) \leqslant 0$. Then, if $\varphi_1(x) \to u_- < 0$ as $x \to -\infty$ and $\varphi_2(x)$ is bounded, then

$$\lim_{t\to\infty} s(t) = +\infty \quad \text{and} \quad \lim_{t\to\infty} u_1(x,t) = U_-.$$

As in Theorem 3, it is enough to prove that for any $\varepsilon_1 > 0$ the inequality $u_2(x,t) \leqslant \varepsilon_1$ holds for $s(t) < x < k\sqrt{t} + N$ and $t \geqslant T$, where $k > 0$ is a sufficiently large number and $N > 0$. For this purpose construct the functions $w_\pm(x,t) = Me^{(\theta x - \gamma t)} \pm u_2(x,t)$. Then

$$L_2(w_\pm) = M(\theta^2 + \gamma A_2 + B_2\theta + C_2)\times$$
$$\times\, e^{(\theta x - \gamma t)} \leqslant 0,$$

$w_\pm(x,0) = Me^{\theta x} \pm \varphi_2(x) \geqslant 0$ for $x \geqslant 0$, and $w_\pm(x,t)|_{x=s(t)} \geqslant 0$ for sufficiently small θ, γ and large M. Hence we obtain $|u_2(x,t)| \leqslant Me^{(\theta x - \gamma t)}$. Consequently, $u_2(x,t) \leqslant \varepsilon_1$ for $s(t) < x < k\sqrt{t} + N$ and $t \geqslant T$. This theorem is also valid in the case when $A_2(x,t)$ is only bounded.

Theorem 5. Let $u_i(x,t)$ be a solution of problem (1)–(3), and let the coefficients $A_1(x,t) \to a_1^2$, $B_1(x,t) = o(1/\sqrt{t})$ as $t \to \infty$, uniformly in x, with $xB_i(x,t) \geqslant 0$ for $|x| \geqslant x_0$, $C_1 = 0$, $C_2 \leqslant 0$. Then, if $\varphi_1(x) \to u_- < 0$ as $x \to -\infty$ and $\varphi_2(x) \to 0$ as $x \to +\infty$, then $\lim_{t\to\infty} s(t) = \infty$ and $\lim_{t\to\infty} u_1(x,t) = U_-$.

Theorem 6. For the solution of problem (1)–(3) the following assertions are valid: 1) $\lim_{t\to\infty} s(t) = +\infty$, $\lim_{t\to\infty} u_1(x,t) = u_-$, if $C_1 = 0$, $B_i(x,t) \leqslant -b < 0$, $\varphi_1(x) \to u_-$ as $x \to -\infty$; 2) $\lim_{t\to\infty} s(t) = -\infty$, $\lim_{t\to\infty} u_2(x,t) = u_+$, if $C_2 = 0$, $B_i(x,t) \geqslant b > 0$, $\varphi_2(x) \to u_+$ as $x \to \infty$; 3) $\lim_{t\to\infty} u_i(x,t) = 0$, if $C_i(x,t) \leqslant -C < 0$ or $B_1(x,t) \geqslant b_1 > 0$, $B_2(x,t) \leqslant -b_2 < 0$.

The proofs of Theorems 5 and 6 are similar to the proofs of the preceding theorems.

One may consider a generalized solution of problem (1)–(3). To this end we write (1)–(2) in the form $u_{xx} + B(x,t,u)u_x + C(x,t,u) = A(x,t,u)u_t$ and $u(x,0) = u_0(x)$. A continuous bounded function $u(x,t)$ will be called a generalized solution of problem (1)–(3) if, for every continuously differentiable function $f(x,t)$ equal to zero outside some finite domain, $u(x,t)$ satisfies the equality

$$\iint_{t\geqslant 0}\left[u\frac{\partial^2 f}{\partial x^2} - \overline{B}(x,t,u)\frac{\partial f}{\partial x} + G(x,t,u)f + \Phi(x,t,u)\frac{\partial f}{\partial t}\right]dx\,dt - \int_{-\infty}^{\infty}\Phi(x,0,u_0(x))f(x,0)\,dx = 0,$$

where

$$\Phi = \int_0^u A(x,t,u)\,du \quad \text{for } u<0, \qquad \Phi = \int_0^u A\,du + 1 \quad \text{for } u>0,$$

$$\overline{B} = \int_0^u B(x,t,u)\,du \quad \text{and} \quad G(x,t,u) = C(x,t,u) + \int_0^u \frac{\partial A}{\partial t}\,du - \int_0^u \frac{\partial B}{\partial x}\,du.$$

If one assumes that $u_0(x)$ has a generalized derivative summable with its square, and that $C_1(x,t) = C_2(x,t) = C(x,t)$, then, as in work (5), one can prove that there exists a unique generalized solution of problem (1)–(3). This solution can be obtained as the limit, as $h \to 0$, of a uniformly convergent sequence of solutions of the Cauchy problem for the equation $u_{xx} + B^h(x,t,u^h)u_x^h + C(x,t)u^h = \Phi_u^h(x,t,u^h)u_t^h$ with the condition $u^h(x,0) = u_0^h(x)$, where B^h and Φ^h are the mean functions of B and Φ, respectively, and $u_0^h(x)$ is the mean function of $u_0(x)$ (see (6)). It can be proved that Theorem 2 for $A_i(x,t) \equiv a_i^2$, $B_i \equiv 0$, Theorems 4 and 5 for $C_2(x,t) \equiv 0$, and Theorem 6 for $C_1(x,t) = C_2(x,t)$ are valid also for the generalized solution of problem (1)–(3).

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References

1. L. I. Rubinshtein, DAN, 77, No. 1 (1958).

2. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Moscow, 1953, p. 262.

3. A. M. Il' in, UMN, 16, issue 2 (1961).

4. A. M. Il' in, O. A. Oleinik, DAN, 120, No. 1 (1958).

5. O. A. Oleinik, DAN, 135, No. 5 (1960).

6. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950, p. 19.

Note: Figure translations are in progress. See original paper for figures.