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Abstract

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RADICALS IN GROUP THEORY

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In recent years numerous works have been published devoted to the study of groups close to abelian groups or close to finite groups. Numerous classes of groups have been introduced and continue to be introduced, relations between them are studied, and an enormous amount of material has been accumulated, already with difficulty susceptible to survey and systematization. The concept of an abstract group property does not save the situation, since to any class of groups there corresponds some property.

In the present note it will be shown that, for arranging the material accumulated up to the present time, the axiomatic concept of a radical can be used successfully. It should be taken here in the form indicated in the author' s paper ([1]), and not in the modification that was indicated later by B. I. Plotkin ([2]). In ([1]) radicals in rings and algebras were considered, but all constructions are immediately transferred to any class of multioperator groups; in the special case of groups simply, additional circumstances arise by virtue of which many known classes of groups turn out to be radical or semisimple classes with respect to certain radicals.

1. A class of groups R is called **radical** if:

I. 1. A homomorphic image of an R-group is an R-group.

I. 2. Every group has an R-**radical**, i.e. an invariant R-subgroup containing all its other invariant R-subgroups.

I. 3. The factor group of every group by its R-radical is R-**semisimple**, i.e. contains no invariant R-subgroups distinct from E.

Radical and semisimple classes mutually determine one another: a group is R-radical if and only if it is not mapped homomorphically onto a nonidentity R-semisimple group. The R-radical of every group is its characteristic subgroup. By property I.1 every group has a unique representation as an extension of an R-radical group by means of an R-semisimple group.

A class of groups R is radical if and only if at least one of the conditions is satisfied:

II. 11. A homomorphic image of an R-group is an R-group.

III. 12. Every nonidentity homomorphic image of an R-group contains a nonidentity characteristic R-subgroup.

And also at least one of the conditions:

II. 21. Every group, any nonidentity homomorphic image of which possesses a nonidentity invariant R-subgroup, is itself an R-group.

III. 22. Every group possessing an ascending invariant series with R-factors is itself an R-group.

IV. 23. Every group possessing an ascending invariant series whose factors are generated by their characteristic R-subgroups is itself an R-group.

It follows from this that a radical class is closed with respect to extensions and with respect to direct products.

2. Let the class of groups M be closed with respect to homomorphic images. By $R_0(M)$ we shall denote the minimal radical class containing M. Groups from M will be called groups of the 1st degree over M. A group G is a group of the β-th degree over M, where β is an ordinal number, if every nonidentity homomorphic image of it contains a nonidentity invariant subgroup of the α-th degree for some $\alpha < \beta$.

The radical class $R_0(M)$ consists of all groups having some degree over M.

The radical class $R_0(M)$ consists of those and only those groups which possess an ascending normal series with M-factors, such that its members are attainable along this series from the group itself in a finite number of steps.

We note that normal series of this kind were considered in the author' s paper (3), where Schreier' s and Jordan–Hölder' s theorems were generalized to them.

Under the same condition on M as above, the class $R(M)$ of groups possessing an ascending normal series with M-factors is radical.

3. *A class of groups S is semisimple if and only if at least one of the following conditions is satisfied:*

III. 11. *A normal divisor of an S-group is an S-group.*

IV. 12. *Every nonidentity normal divisor of an S-group maps homomorphically onto a nonidentity S-group.*

And also at least one of the following conditions:

III. 21. *Every group any nonidentity normal divisor of which maps homomorphically onto a nonidentity S-group is itself an S-group.*

IV. 22. *Every group possessing a descending normal (invariant, characteristic) series whose factors are approximated by S-groups (i.e. are subdirect products of S-groups) is itself an S-group.*

V. 23. *Every group possessing a descending normal series with S-factors is itself an S-group.*

From this follows the closure of a semisimple class with respect to extensions and subdirect products.

4. Let the class of groups N be closed with respect to normal divisors. By $S_0(N)$ we shall denote the minimal semisimple class containing N.

The semisimple class $S_0(N)$ consists of those and only those groups every nonidentity normal divisor of which maps homomorphically onto a nonidentity N-group.

The semisimple class $S_0(N)$ consists of those and only those groups which possess a descending normal (invariant, characteristic) series whose factors are approximated by N-groups.

The semisimple class $S_0(N)$ consists of those and only those groups which possess a descending normal series with N-factors.

Under the same condition on N as above, the class $S(N)$ of groups possessing a normal system with N-factors will be semisimple.

5. We see that many classes of generalized soluble groups (M or N the class of abelian groups) are radical or semisimple. Among them the minimal semisimple class will be the class of groups with a descending series of commutants. The closure of the class of groups with a soluble normal system with respect to approximation (see (4)) follows from its semisimplicity. *The minimal radical soluble class is distinct from the corresponding class* $R(M)$ (i.e. from the class RN^*): O. Yu. Schmidt's example (5) of a locally finite p-group containing no abelian normal divisors contains no abelian attainable subgroups either.

The question of the radicality of the class RI^*, i.e. of its coincidence with the corresponding class $R_0(M)$, remains open. In the general case, for a class M closed with respect to homomorphic images, the class of groups possessing an ascending invariant series with M-factors is the class of groups of the 2nd degree over M. If M is the class of all cyclic groups, then this will be the class

supersoluble groups (⌒6); it is not radical. The case where M or N is the class of all finite groups includes, in particular, the works of Specht (⌒7) and A. I. Mal' cev (⌒8).

The class of periodic groups with collectively bounded orders of elements generates a semisimple class. The corresponding radical groups are complete in the sense of S. N. Chernikov. Question 1.4 from the survey of S. N. Chernikov (⌒9) is the question of distinguishing this radical from the radical whose semisimple class consists of all groups containing no complete abelian subgroups. *A positive answer to this question is given by the group with generators*

$$a_{nk}, \quad k = 1, 2, \dots, 2^{n-1}, \quad n = 1, 2, \dots,$$

and defining relations (for all n and k)

$$a_{nk} = a_{n+1,\,2k-1}^{(n+1)!}\, a_{n+1,\,2k}^{(n+1)!}.$$

6. Radicals may be considered in any class of groups closed under homomorphic images and normal divisors. *In the class of p-primary abelian groups there exists a unique nontrivial radical:* the radical groups are the complete groups, and the semisimple groups are the reduced ones. If this semisimple class is regarded as generated by all cyclic groups of orders p^n, $n = 1, 2, \ldots$, then the Ulm series of a reduced group is the decreasing characteristic series whose factors are approximated by cyclic groups. Radicals in the class of all periodic abelian groups are readily surveyed.

In the class of all finite groups, the radical classes generated by all possible sets of simple groups, and only they, are closed under normal divisors, i.e. are also semisimple classes. Such are, in particular, the classes of Π-soluble and Π-separable groups introduced by S. A. Chunikhin (^10).

The study of radicals in classes of groups closed under homomorphic images but not under normal divisors (see, for example, the complete radical of V. I. Ushakov (^11) in the class of torsion-free groups) will require considering only those normal divisors whose factor groups belong to the class under consideration.

A reconsideration, from the point of view of the concept of radical, of all the material accumulated in group theory will undoubtedly lead to a serious restructuring of this theory and will open many new paths for further research.

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Note: Figure translations are in progress. See original paper for figures.