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Abstract

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ON NONHOLONOMIC CONGRUENCES W

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It is known (^{1,2}) that a single not completely integrable Pfaff equation with respect to the three principal parameters of a linear complex singles out in it a set of ruled surfaces, called a nonholonomic congruence of the complex. Many concepts and results of the theory of ordinary (holonomic) congruences of three-dimensional space can be extended to these nonholonomic congruences. One of the most remarkable classes among the latter is formed by the congruences W (^{3}). In this note the nonholonomic analogue of congruences W is considered within the framework of the equiaffine theory of complexes (^{2}), although part of the results has, as is easy to see, a projective-invariant character. The terms from the theory of complexes correspond to those adopted in (^{2,8,9}), and the terms of nonholonomic geometry to those adopted in (^{1,7}).

§ 1. Assign the linear complex to a net consisting of an arbitrary nonholonomic congruence $\omega^1 = 0$ and of the family of ruled surfaces conjugate (^{2,4}) to it. To this net there corresponds a semicanonical frame R' (^{2}), in which the vector e_3 is directed along the ray of the complex, the vector e_1 is parallel to the affine normal of the fundamental cylindroid (i.e., of that cylindroid whose directrix plane is parallel to the tangent plane π of the cylinder of the complex) at the affine center (^{2,5}) of the ray, the points A and $A' = A - e_3$ are foci, and the planes $\{e_2 e_3\}$ and $\{e_1 - e_2, e_3\}$ are focal planes (^{1}) of the nonholonomic congruence. The derivation formulas of this frame have the form:

$$dA = \omega_0^i e_i, \qquad de_i = \omega_i^k e_k \quad (i, k = 1, 2, 3), \tag{1}$$

where

$$\omega_0^2 = \omega_3^1 + \omega_3^2, \qquad \omega_1^2 = A\omega^1 + B\omega_3^1 + C\omega_3^2,$$

$$2\omega_1^1 + \omega_1^2 = -(B\omega^1 + E\omega_3^1 + F\omega_3^2),$$

$$\omega_3^3 - \omega_2^1 - \omega_2^3 = C\omega^1 + F\omega_3^1 + G\omega_3^2,$$

$$\omega_\alpha^3 = \xi_\alpha \omega^1 + \eta_\alpha \omega_3^1 + \zeta_\alpha \omega_3^2 \quad (\alpha = 0, 1, 2, 3),$$

$$B^2 = AE, \qquad \xi_1 = AF - BC.$$

The coefficients of these formulas are invariants of the nonholonomic congruence and have a simple geometric meaning. For example: B is the mean affine curvature of that ruled surface of the nonholonomic congruence whose asymptotic plane is parallel to the plane π; $2C$ is the affine angular coefficient of the tangent to the curvilinear asymptotic at the point A' of the ruled surface described by the straight line

$$r = A' + \lambda e_1$$

under motion along the torsus of the nonholonomic congruence having its focus at A'; $\frac{1}{2}E$ is the affine angular coefficient of the line of intersection of the cone of curvature of the nonholonomic congruence (this cone is defined in the same way as the Haag cone of curvature (^{6}) for a holonomic congruence) with the plane π. The geometric meaning of the remaining invariants A, F, G of the same dif-

differential neighborhood is obtained from the formulas

$$\mathbf{C} = \mathbf{A} + \{(B - A) : 2A\}\mathbf{e}_3,$$

$$At^2(1+t)^2 - 2Bt(1+t)^2 + 2Ct(1+t) + E(1+t)^2 - 2F(1+t) - G = 0, \quad (2)$$

where $\mathbf{C}$ is the affine center, and $\mathbf{I} = \mathbf{A} + t\mathbf{e}_3$ are the inflection centers $^{(8)}$ of the ray of the complex.

§ 2. We shall call a nonholonomic congruence W such a nonholonomic congruence on whose nonholonomic focal surfaces there correspond asymptotic lines. Let us note that holonomic congruences of a complex, as well as nonholonomic congruences with degenerate or parabolic $^{(7)}$ nonholonomic focal surfaces, do not fall under this definition. The equations of the asymptotic lines on the nonholonomic surfaces $\{A\}$ and $\{A'\}$ have, respectively, the form:

$$\begin{aligned} \omega^1 = (\eta_3 - F)(\omega_3^1)^2 + (\zeta_3 + \eta_3 - \eta_0 - F - G)\omega_3^1\omega_3^2 + \\ +(\zeta_3 - \zeta_0 - G)(\omega_3^2)^2 = 0, \end{aligned} \quad (3)$$

$$\omega^1 = (\eta_3 + F - E)(\omega_3^1)^2 + (\zeta_3 + \eta_3 - \eta_0 - F + G)\omega_3^1\omega_3^2 + (\zeta_3 - \zeta_0)(\omega_3^2)^2 = 0.$$

Taking into account that the condition of holonomicity of the congruence $\omega^1 = 0$ is the equality $F - G + \eta_0 - \eta_3 + \zeta_3 - 2\zeta_0 = 0$, we obtain that the nonholonomic congruence $\omega^1 = 0$ is a nonholonomic congruence W if and only if

$$E = 2F, \qquad G = 0. \tag{4}$$

Using formulas (1)–(4) and taking into account the results of $^{(8,9)}$, we obtain the following theorems.

Theorem 1. Both foci of a nonholonomic congruence are inflection centers of the ray of the complex if and only if it is a nonholonomic congruence W.

Theorem 2. In a complex with four distinct inflection centers there are six nonholonomic congruences W; in a complex with one double inflection center there is only one nonholonomic congruence W, and its invariants are connected by the relation $8C = A + 4B$; the remaining complexes contain no nonholonomic congruences W.

Theorem 3. A nonholonomic congruence W has an osculating linear complex, defined in the same way as for a holonomic congruence W; this linear complex is one of the three principal linear complexes (a principal linear complex $^{(2)}$ contains the entire first differential neighborhood of the ray of the complex and the second neighborhood of one of the principal ruled surfaces of the complex); its affine axis is parallel to the line of intersection of the curvature cone with the plane π.

Theorem 4. A nonholonomic congruence W is characterized by the fact that each of the parabolic nonholonomic congruences of the complex whose unique focus coincides with one of the foci of the nonholonomic congruence W has a parabolic focal nonholonomic surface (these two parabolic nonholonomic congruences have, respectively, the equations $\omega^1 = \omega_3^1$ and $\omega^1 + \omega_3^1 + \omega^2 = 0$).

§ 3. We now refer the complex to the canonical frame R $^{(2)}$, consisting of the affine center $\mathbf{B}$ of the ray and the vectors $\mathbf{E}_i$ of the canonical point frame of the principal cylindroid at this center. For this frame we have the derivative formulas:

$$d\mathbf{B} = \Omega^i \mathbf{E}_i, \qquad d\mathbf{E}_i = \Omega_i^k \mathbf{E}_k \quad (i, k = 1, 2, 3), \tag{5}$$

where

$$\mathbf{E}_1 = \mathbf{e}_1, \qquad \mathbf{E}_3 = \mathbf{e}_3, \qquad 2\Omega_1^1 = -v\Omega_3^2,$$

$$\Omega_1^2 = \Omega^1 + z\Omega_3^2, \qquad \Omega_2^1 + \Omega^3 = -(z\Omega^1 + v\Omega_3^1 + w\Omega_3^2)$$

and z, v, w are invariants of the second differential neighborhood of the complex. The nonholonomic congruence $\Omega^1 = \xi\Omega^1_3 + \eta\Omega^2_3$ is a nonholonomic congruence W if and only if the functions ξ and η are connected by the relations

$$\xi^3 + 2\xi^2\eta + 2z\xi + 2v = \xi^2\eta + \eta^2 + 2z\eta + w = 0, \tag{6}$$

where the center of the ray of the nonholonomic congruence is the point $C = B - \frac{1}{2}\xi E_3$. Hence we have:

$$\xi^6 + 4z\xi^4 + 4(z^2 - w)\xi^2 - 4v^2 = 0. \tag{7}$$

Using formulas (5)–(7) and the formulas for the transition from the frame R to the frame R', we obtain the following theorems:

Theorem 5. The centers of nonholonomic congruences W are situated on a ray of the complex in pairs symmetric with respect to the affine center of the ray; moreover, each such pair of nonholonomic congruences W has a common osculating linear complex with an affine axis, whose angular coefficient is $\frac{1}{2}\xi^2$.

Theorem 6. In an affinely symmetric complex $v = 0,\ w \neq z^2$ [2], two nonholonomic congruences W have a common center, coinciding with the affine center of the ray of the complex; their common osculating linear complex has as affine axis the vector $E_1 = e_1$.

Theorem 7. In a complex $w = 0$, three nonholonomic congruences have one of the foci at the affine center of the ray; the invariants of each of the three other nonholonomic congruences W of this complex are connected by the relation $8AC = A^2 - B^2 + 4AB$.

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