MATHEMATICS

A. A. TALALYAN

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Abstract

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A. A. TALALYAN

ON THE EXISTENCE OF A TRIGONOMETRIC SERIES UNIVERSAL WITH RESPECT TO SUBSERIES

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In 1941 D. E. Men' shov (^1) proved the following fundamental theorem on the representation of measurable functions by trigonometric series:

For every almost everywhere finite measurable function $f(x)$, defined on $[-\pi, \pi]$, there exists a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

which converges to $f(x)$ almost everywhere on $[-\pi, \pi]$.

In proving this theorem, D. E. Men' shov used the following theorem, proved by him in paper (^2):

For every almost everywhere finite measurable function $f(x)$, defined on $[-\pi, \pi]$, and for every positive number σ, one can determine a function $G(x)$, continuous on $[-\pi, \pi]$, and a measurable set E, possessing the following properties:

$$\alpha)\ E \subset [-\pi, \pi], \quad \operatorname{mes} E > 2\pi - \sigma;$$

$$\beta)\ G(x) = f(x) \quad (x \in E);$$

γ) the Fourier series of the function $G(x)$ converges almost everywhere on $[-\pi, \pi]$.

Using Men'shov's lemmas proved in paper (^2), and applying a method different from the method used in (^1), one can prove the following theorem, which is a strengthening of Men' shov' s theorem on the representation of measurable functions by trigonometric series:

Theorem 1. There exists a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \tag{1}$$

such that for every almost everywhere finite measurable function $f(x)$, defined on $[-\pi, \pi]$, there exists a subseries of the series (1)

$$\sum_{k=1}^{\infty} a_{n_k} \cos n_k x + b_{n_k} \sin n_k x \quad (n_1 < n_2 < \cdots < n_k < \cdots), \tag{2}$$

which converges to $f(x)$ almost everywhere on $[-\pi, \pi]$.

This theorem is a positive solution of the question posed in the survey article (̂3) (see (̂3), § 10, problem 3).

We shall outline the proof of Theorem 1 in general terms.

We construct a series (1) having the following properties (we formulate these properties as lemmas, because they may be regarded as properties of trigonometric polynomials in general).

Lemma 1. *For any almost everywhere finite measurable function $\psi(x)$, given on some set E, $\operatorname{mes} E > 0$, for any natural number N and positive number δ, from the terms of the series (1) one can form a trigonometric polynomial*

$$\sum_{j=1}^{q} \left(a_{p_j} \cos p_{jx} + b_{p_j} \sin p_{jx} \right)$$

and determine measurable sets P and Q such that the following conditions are satisfied:

$$N < p_1 < p_2 < \cdots < p_q; \tag{3}$$

$$P \subset E, \qquad \operatorname{mes} P > \operatorname{mes} E - \delta; \tag{4}$$

$$Q \subset CE, \qquad \operatorname{mes} Q > \operatorname{mes} CE - \delta \qquad (CE = (-\pi, \pi) - E); \tag{5}$$

$$\left| \sum_{j=1}^{q} a_{p_j} \cos p_{jx} + b_{p_j} \sin p_{jx} - \psi(x) \right| < \delta \tag{6}$$

for $x \in P$;

$$\left| \sum_{j=1}^{i} a_{p_j} \cos p_{jx} + b_{p_j} \sin p_{jx} \right| < \delta \tag{7}$$

for $x \in Q$ and for all i, where $1 \leq i \leq q$.

Lemma 2. *For any positive number ε one can determine a positive number δ such that, whatever the measurable function $f(x)$ defined on some measurable set $E \subset (-\pi, \pi)$, $\operatorname{mes} E > 0$, and satisfying the inequality*

$$|f(x)| \leq \delta, \qquad x \in E, \tag{8}$$

and whatever the natural number N and positive number η, there exist a measurable set F and a polynomial

$$\sum_{j=1}^{r} a_{k_j} \cos k_{jx} + b_{k_j} \sin k_{jx},$$

chosen from the series (1), which satisfy the following conditions:

$$F \subset E, \qquad \operatorname{mes} F > \operatorname{mes} E - \varepsilon; \tag{9}$$

$$N < k_1 < k_2 < \cdots < k_r; \tag{10}$$

$$\left| \sum_{j=1}^{\rho} a_{k_j} \cos k_{jx} + b_{k_j} \sin k_{jx} \right| \leq \varepsilon \tag{11}$$

for any $x \in F$ and for all ρ, where $1 \leq \rho \leq r$;

$$\left| \sum_{j=1}^{r} a_{k_j} \cos k_{jx} + b_{k_j} \sin k_{jx} - f(x) \right| > \eta \tag{12}$$

for any $x \in F$.

It is further proved that the series (1), constructed in such a way that Lemmas 1 and 2 hold for it, satisfies the conditions of Theorem 1.

In constructing the series (1) possessing the indicated properties, the following lemma of D. E. Menshov is used (2):

Lemma 3 (D. E. Menshov). *Let $ax + b$ be a linear function which does not change sign on some interval $[c, d]$.*

Then, for any positive number η and integer $\nu > 8$, one can define a function $\psi(x)$ and a measurable set E that have the following properties:

1) $\psi(x)$ is continuous on $[c, d]$ and is linear on each of a certain finite number of intervals obtained by partitioning the interval $[c, d]$;
2) $\psi(c) = ac + b$, $\psi(d) = ad + b$;
3) $E \subset [c, d]$, $\operatorname{mes} E > (d - c)\left(1 - \dfrac{8}{\nu}\right)$;
4) $|\psi(x)| \leqslant 2\varkappa\nu$ $(c \leqslant x \leqslant d)$, where $\varkappa = \max\limits_{c \leqslant x \leqslant d} |ax + b|$;
5) $$\left| \int_{c'}^{d'} \psi(t)\, dt \right| < \eta$$

   for all c' and d', where $c \leqslant c' < d' \leqslant d$;

6)
$$\left|\int_c^d \psi(t)\frac{\sin n(t-x)}{t-x}\,dt\right| \leqslant B\varkappa\nu$$

$(x \in E,\ n = 1, 2, ...)$, where B is an absolute constant;

7) $\psi(x) = ax + b$ for $x \in E$.

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

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Note: Figure translations are in progress. See original paper for figures.