ON THE APPROXIMATION OF FUNCTIONS OF MANY VARIABLES WITH PRESERVATION OF BOUNDARY CONDITIONS

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Abstract

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ON THE APPROXIMATION OF FUNCTIONS OF MANY VARIABLES WITH PRESERVATION OF BOUNDARY CONDITIONS

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1. Let D be an m-dimensional domain with boundary Γ, and let $\varphi(x) = 0$ be the equation of Γ. We consider the approximation of functions $u(x)$, equal to zero on Γ together with derivatives up to order $s-1$, by expressions of the form $\varphi^s(x_1, x_2, \dots, x_m)P_n(x_1, x_2, \dots, x_m)$, where $P_n(x_1, x_2, \dots, x_m)$ is a polynomial of degree not exceeding n in each of the arguments $x_1, x_2, \dots, x_m$. This problem, connected with variational methods for solving boundary-value problems for elliptic equations, was considered in papers ([1–3]).

The results set forth below are based essentially on a certain new method of extending a function with preservation of differential properties from a domain whose boundary contains singular points of a definite kind.

Denote by $C^k_\omega(D)$ the space in which the unit sphere is the set of functions f having in the domain D continuous partial derivatives up to order k, with

$$\max_{0\le l\le k}\max_{x\in\overline{D}}|D^l f(x)| \le 1, \qquad \omega_{C(D)}(D^l f;\varepsilon) \le \omega(\varepsilon)$$

$$(0 \le l \le k).$$

If $\omega(\varepsilon) = \varepsilon$, then we shall use the notation $C^{k,1}(D)$. By $\widetilde{C}^k_\omega$ and $\widetilde{C}^{k,1}_\omega$ we denote the corresponding spaces of functions that are 2π-periodic in all arguments.

Under certain restrictions on the function $f(x)$ the following holds.

Proposition 1. *For every function $u(x)$ from $C^k_\omega(D)$, equal to zero on the boundary together with derivatives up to order $s-1$ inclusive $(0 \le s-1 \le k)$, there exists a sequence of polynomials $P_n(x)$ of degree not exceeding n in each variable, for which*

$$\|u - \varphi^{sP}_n\|_{C^l(D)} \le N_1\|u\|_{C^k_\omega(D)}\, n^{-(k-l)}\omega(1/n),$$

where $l = 0, 1, \dots, k$, and N_1 does not depend on n or on u.

In paper ([3]) it was proved:

Theorem 1. *Proposition 1 is valid if* $\varphi(x)$ *is such that: a)* $\varphi(x) \in C^{k,1}(D)$*; b)* $\varphi(x) > 0$ *in* D*; c) for each point* y *of the boundary* Γ *there exists a number* $p = p(y)$ *and a neighborhood* Ω_y*, in which the function* $\varphi(x)$ *is representable in the form* $\varphi(x) = \varphi_1(x)\varphi_2(x)\cdots\varphi_p(x)$*, the functions* $\varphi_i(x)$ *satisfying the conditions: 1)* $\varphi_i(x) > 0$ *in* $\Omega_y \cap D$*; 2)* $\varphi_i(x) \in C^{k,1}(\Omega_y)$*; 3)* $\varphi_i(y) = 0$*; 4)* $\operatorname{grad}\varphi_i(y) \neq 0$*; d) the vectors* $\operatorname{grad}\varphi_i(y)$ $(i = 1, 2, \dots, p)$ *are linearly independent.*

In papers ([1,2]) an analogous result was established under the assumption that $p(y) = 1$ at every point y of the boundary Γ, for the cases $s = 1$ and $s = 2$.

It follows from the conditions of Theorem 1 that the boundary Γ, in a sufficiently small neighborhood of an arbitrary point y of it, is formed by the intersection of p surfaces whose normals are linearly independent. In the present note the class of domains and functions $\varphi(x)$ for which Proposition 1 is valid is enlarged. For some domains, for which the theorem is certainly false, results of an analogous kind are established.

2. Everywhere below we assume that the functions $\varphi(x)$ satisfy conditions a) and b) of Theorem 1.

We shall call a boundary point x_0 of the domain D **admissible for the function** $\varphi(x)$ if there exists a neighborhood Ω_{x_0} such that Proposition 1 is valid for the given D and $\varphi(x)$ for every function $u(x)$ satisfying the additional requirement: $u(x) \equiv 0$ outside Ω_{x_0}.

If all boundary points are admissible for the function $\varphi(x)$, then Proposition 1 is valid for it. In particular, any point y at which conditions c) and d) of Theorem 1 are fulfilled for the function $\varphi(x)$ is admissible. From condition d) it follows that $p \leq m$, so that, for example, Theorem 1 does not include among the domains under consideration a polyhedron in three-dimensional space at some vertex of which more than three faces meet. This restriction is removed by the following

Theorem 2. *If at a boundary point* y *condition c) of Theorem 1 is fulfilled, and* $p > m$*, but any* m *of the vectors* $\operatorname{grad}\varphi_i(y)$ $(i = 1, 2, \dots, p)$ *are linearly independent and* $k \geq ps$*, then the point* y *is admissible.*

The proof is carried out as in ([3]). The main difficulty lies in the proper extension of the function $u(x)$ from the domain $D \cap \Omega_y$ to the whole neighborhood Ω_y, namely in such a way that the extended function is equal to zero together with its derivatives up to order $s - 1$ on the surfaces $\varphi_i(x) = 0$ $(i = 1, 2, \dots, p)$. If for $p \leq m$ it was sufficient for this purpose to apply Hestenes' method (see ([4])), here one has to use more delicate arguments. We shall need a number of lemmas.

Lemma 1. *If an* m*-dimensional domain* D *satisfies the condition: any two of its points* x, y *can be joined by a curve lying in the domain of length* $l \leq \lambda(x, y)$*, where* $|xy|$ *is the distance between the points and* λ *does not depend on* x *and* y*,* *then any function $f(x)$ from $C^k_\omega(D)$ can be extended to the whole space E_m with preservation of smoothness, and moreover*

$$\|f\|_{C^k_\omega(E_m)} \leq N_2 \|f\|_{C^k_\omega(D)},$$

where N_2 depends only on m, k, λ.

The proof of this lemma is obtained by a certain detailing of Whitney' s method ([5]), simplified by Hestenes ([6]).

Lemma 2. *Let S be the unit sphere of m-dimensional Euclidean space and $f \in C^k_\omega(S)$, and suppose that at the origin $D^i f = 0$, $i = 0, 1, \ldots, k$; let the behavior of the function $\chi(x)$ at the origin be determined by the estimate*

$$|D^l \chi(x)| \leq N_3(l)|x|^{-(l+\alpha)},$$

where $|x|$ is the distance of the point x from the origin, $l = 0, 1, 2, \ldots$, and α is some integer, $0 \leq \alpha \leq k$. Then the function $f_1(x) = f(x)\chi(x)$ belongs to the class $C^{k-\alpha}_\omega(S)$, and

$$\|f\chi\|_{C^{k-\alpha}_\omega(S)} \leq N_4 \|f\|_{C^k_\omega(S)},$$

where N_4 does not depend on f.

Lemma 3. *Let Δ_1, Δ_2 be domains on the surface Σ of the sphere S from Lemma 2 whose closures do not intersect; let $\lambda(x)$ be a function given on the surface Σ, infinitely differentiable, equal to 0 in Δ_1 and equal to 1 in Δ_2. Then the function*

$$\chi(x) = \lambda\left(\frac{x}{|x|}\right)$$

satisfies the condition

$$|D^l(\chi(x))| \leq N_5 |x|^{-l},$$

where N_5 depends only on l and on the function $\lambda(x)$.

Lemma 4. *Under the conditions of Theorem 2, the function $u(x)$ from Proposition 1 is representable in the form $u(x) = \varphi^s q(x) + u_1(x)$, where $q(x)$ is a polynomial of degree $k - ps$, and the function $u_1(x)$ at the point y is equal to zero together with all derivatives up to order k inclusive.*

The principal part of the function $u(x)$ at the point y singled out in Lemma 4, i.e. $\varphi^s q$, has already been defined in the whole neighborhood Ω_y; it remains only to extend the function $u_1(x)$. For this it suffices to extend it from the domain D, while preserving smoothness and the zero conditions on the surfaces $\varphi_i(x) = 0$ $(1 \leq i \leq p)$, to a function $\bar{u}_1(x)$ defined in some conical neighborhood Λ of the domain D (this can be done by Hestenes' method), choose an intermediate conical neighborhood Λ_1 $(\bar{D} \subset \Lambda_1 \subset \bar{\Lambda}_1 \subset \Lambda)$, construct by Lemma 3

the function $\chi(x)$, equal to 1 inside Λ_1 and equal to 0 outside Λ, and then set $u_2 = \overline{u}_1\chi$ inside Λ and $u_2 \equiv 0$ outside Λ. Applying Lemma 2, we see that the smoothness of the function u_2 is the same as that of u_1, while the satisfaction of the zero conditions on the surfaces $\varphi_i(x)$ is obvious. Consequently, the function $\overline{u}(x) = \varphi(x)q(x) + u_2(x)$ will be the required extension.

Theorem 3. *Suppose that in a neighborhood of a point y the function $\varphi(x)$ is such that, under a smooth change of coordinates $x = \sigma(\xi)$ (the $x_i(\xi)$ are functions of class $C^{k+1,1}$, $\xi_i(y) = 0$, and the Jacobian $|\sigma(\xi)|$ is different from zero), it becomes the function*

$$\hat{\varphi}(\xi) = \varphi(\sigma(\xi)) = \xi_m^2 - (\xi_1^2 + \xi_2^2 + \cdots + \xi_{m-1}^2),$$

and, in the domain $\hat{D} \cap \hat{\Omega}_y$, we have: $\hat{\varphi}(\xi) > 0$, $\xi_m > 0$. If $k \geq 2s$, then for the function $\varphi(x)$ the point y is admissible.

For the proof, first, as in Theorem 1, we extend the function $\hat{u}(\xi) = u(\sigma(\xi))$, defined only in that half of the cone where $\hat{\varphi}(\xi) > 0$, $\xi_m > 0$, to the whole neighborhood $\hat{\Omega}_y$ by the method of separating the principal part. However, unlike Theorem 2, here one cannot directly use the method of the paper ([3]). True, the polynomials $P_n(x)$ are constructed from the function $u(x)$ as in ([3]): extending u and φ in the proper way to the cube $|x_i| \leq a$ containing $\bar{D}$, we form the function $v = \varphi^{-s}u$; we pass to periodic functions $\tilde{u}, \tilde{\varphi}, \tilde{v}$, setting $x_i = a\cos\xi_i$, $i = 1, 2, \ldots, m$; finally, we set

$$\widetilde{P}_n(\xi_i) = \tilde{v} - (I - T_n)^k\tilde{v},$$

where T_n is an integral operator with a Jackson-type kernel. But in order to prove that the polynomials $P_n(x_i) = \widetilde{P}_n(\arccos x_i)$ are the desired ones, the reasoning has to be further complicated. We prove, using the method of separating the principal part and Lemma 2, that for the functions

$$v_{l,p} = \varphi^{-l}(D_{i_1}\varphi D_{i_2}\varphi \cdots D_{i_p}\varphi)\, u,$$

where $0 \leq p \leq l \leq s$, $1 \leq i_r \leq m$, and D_i denotes differentiation with respect to x_i, the estimate

$$\|v_{l,p}\|_{\widetilde{C}_\omega^{k-2l+p}(\Omega_y)} \leq N_6\|u\|_{C_\omega^k(\Omega_y)}$$

is valid, and then the following lemma is applied:

Lemma 5. *Let φ be from $\widetilde{C}^{k+1,1}$, and let u be from $\widetilde{C}_\omega^k$, and suppose these are periodic functions such that the function of the form*

$$v_{l,p} = \varphi^{-l}(D_{i_1}\varphi D_{i_2}\varphi \cdots D_{i_p}\varphi)u,$$

where $0 \le p \le l \le s$, *belongs to the class* $\widetilde{C}_\omega^{k-2l+p}$. *Then the estimate is valid:*

$$\|\varphi^s (I - T_n)^k v_{s,0}\|_{\widetilde{C}^r} \le N_7 A(u)\,\omega(1/n)\,n^{-(k-r)},$$

where

$$A(u) = \max_{0 \le p \le l \le s} \|v_{l,p}\|_{\widetilde{C}_\omega^{k-2l+p}}, \qquad 0 \le r \le k.$$

Theorem 4. *Let* $m = 2$, *and suppose that at a point* y *the boundary* Γ *has the form of a "beak," while the function* $\varphi(x)$ *satisfies condition c) of Theorem 1, but the vectors* $\operatorname{grad}\varphi_1(y)$, $\operatorname{grad}\varphi_2(y)$ *are collinear. Then for the function* $\varphi(x)$ *the point* y *is admissible if* $k \ge 2s$.

The proof is carried out exactly as in Theorems 2, 3. Under the conditions of Theorem 1, the neighborhood Ω_y is divided by the surfaces $\varphi_i(x) = 0$ $(1 \le i \le p)$ into 2^p parts, only one of which belongs to the domain D. If, however, the remaining $2^p - 1$ parts belong to the domain D, then Proposition 1 is false, no matter how the function $\varphi(x)$ is chosen. However, if, except for y, all points of the boundary Γ are admissible, then the following is valid:

Proposition 2. *For every function* $u(x)$ *from* $C_\omega^k(D)$, *equal to zero on the boundary together with derivatives up to order* $s-1$, *there exist sequences* P_n *and* $Q_{n,i}$ *of polynomials for which*

$$\left\| u - \varphi_n^{sP} - \sum_{i=1}^{p} \psi_i Q_{n,i} \right\|_{C^l(D)} \le N_8 \|u\|_{C_\omega^k(D)}\,\omega(1/n)\,n^{-(k-l)},$$

where $l = 0, 1, \ldots, k$; N_8 *does not depend on* u, n; *the function* $\psi_i(x)$ *in the neighborhood* Ω_y *has the form*

$$\psi_i(x) = [\varphi_i(x)]^{k+1} \quad \text{if } \varphi_i(x) \le 0, \qquad \psi_i(x) \equiv 0 \quad \text{if } \varphi_i(x) > 0,$$

and outside Ω_y *it is defined arbitrarily, provided only that it belongs to* $C^{k,1}(D)$ *and vanishes on the boundary together with derivatives up to order* $s-1$.

An analogous proposition is valid if, in the neighborhood Ω_y, the domain D contains not $2^p - 1$, but another number of the mentioned 2^p parts.

Let us also note that the boundary of the domain may contain any set of points of the indicated kind. It is only necessary that the $\psi_i(x)$ have the corresponding representation in a neighborhood of each such point.

4. Let us point out some unsolved questions. In the case $m = 2$ we studied the approximation of functions with preservation of boundary conditions in domains with piecewise smooth boundaries for various values of the angle α between neighboring curves, directed toward the domain D. The case $0 < \alpha < \pi$ corresponds to paper [(3)], and the cases $\alpha = 0$ and $\pi < \alpha < 2\pi$ correspond, respectively, to Theorem 4 and Proposition 2 of the present note. Thus the cases $\alpha = \pi$ and $\alpha = 2\pi$ remain. In the second of these,

the main difficulty lies in the inapplicability of Lemma 1 on the extension of functions. If one restricts oneself to the class of functions u admitting an extension from the domain D with preservation of smoothness, then one can obtain a result of the type of Proposition 2 for a function $\varphi(x)$ satisfying the conditions of Theorem 1, and for appropriately chosen functions $\psi_i(x)$. In the case $\alpha = \pi$, however, not every function $u(x)$ can be extended from the domain D with preservation of smoothness and simultaneous preservation of the zero conditions on the lines $\varphi_1(x) = 0$, $\varphi_2(x) = 0$. Nevertheless, here too one can obtain a result of the type of Proposition 2, but this requires essentially new methods, and the scope of this note does not allow us to dwell on them in greater detail.

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Note: Figure translations are in progress. See original paper for figures.