MATHEMATICS

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1961

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Abstract

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MATHEMATICS

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ON QUASIANALYTIC CONTINUATION OF ANALYTIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 19 VII 1961)

Let E be an arbitrary continuum in the plane P of the complex variable z; let G be a finite or infinite domain not intersecting the continuum E; suppose that a sequence of rational functions $r_n(z)$ of the form

$$r_n(z) = \frac{a_{n0}z^n + a_{n1}z^{n-1} + \cdots + a_{nn}}{(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})}, \qquad \alpha_{nk} \in \overline{E}, \quad \{\alpha_{nk}\}' \cap G = 0, \quad (1)$$

converges uniformly on E to a function $f_1(z)$ (continuous on E and analytic on the set of interior points of E) and inside the domain G to a function $f_2(z)$ (analytic in G):

$$\lim_{n\to\infty} r_n(z) = \begin{cases} f_1(z), & z \in E, \\ f_2(z), & z \in G. \end{cases}$$

We note that no restrictions are imposed on the location of the poles α_{nk} of the rational functions $r_n(z)$ in $P \setminus (E \cup G)$. The question is posed as follows: under what conditions on the rate of convergence of the sequence $r_n(z)$ on the continuum E are the values of the function $f_2(z)$, $z \in G$, uniquely determined by the values of the function $f_1(z)$ on the continuum E, and in this sense the function $f_2(z)$ is a quasianalytic continuation of the function $f_1(z)$; in particular, under what conditions does it follow from $f_1(z) = 0$, $z \in E$, that also $f_2(z) = 0$, $z \in G$.

Some results of this type were obtained, in particular, by Borel ([1]; see also [6]) for functions of the form

$$\sum_{n=1}^{\infty} \frac{A_n}{z - z_n};$$

by S. N. Mergelyan [2] for the case of approximation by polynomials in tangent domains; in a number of cases the solution of the problem posed follows from the results on overconvergence of sequences of rational functions given in [6].

Theorem 1. Let E be an arbitrary continuum; let L_R, $R > 1$, be the image of the circle of radius R with center at 0 under the mapping of the exterior of the unit circle onto the component of the complement of E containing the infinitely distant point; let G be an arbitrary domain having as one of the components of its boundary the level line L_R. Suppose that a sequence of rational functions $r_n(z)$ of the form (1) converges on E to a function $f_1(z)$ and in the domain G to an analytic function $f_2(z)$, and moreover

$$\overline{\lim}_{n\to\infty}\left[\max_{z\in E}|f_1(z)-r_n(z)|\right]^{1/n}=q<\frac{1}{R}.$$

Then, if $f_1(z) = 0$, $z \in E$, then also $f_2(z) = 0$, $z \in G$.

We note that Theorem 1 remains valid if, instead of a sequence of rational functions $r_n(z)$, one considers a sequence of functions of the form $p_n(z)/a_n(z)$, where $p_n(z)$ is a polynomial of degree n, and $a_n(z)$, $n = 1, 2, ...$, is an arbitrary sequence of functions analytic-

functions in a simply connected domain D containing E and G (in particular, an arbitrary sequence of entire functions).

In the theorem one may consider a sequence of rational functions of the form (1) for $n = n_1, n_2, ...$; thereby the sign of the upper limit may be replaced by the sign of the lower limit.

Theorem 1 is sharp in a certain sense; indeed, the sequence of rational functions

$$r_n(z)=\frac{z^n}{z^n-R^n},\qquad R>1,$$

converges to zero in the closed unit disk $|z| \leqslant 1$ and to one in the domain $|z| > R$, and

$$\lim_{n\to\infty}\left[\max_{|z|\leqslant 1}|r_n(z)|\right]^{1/n}=\frac{1}{R}.$$

Corollary 1. *Let E be an arbitrary continuum, D a domain containing this continuum in its interior; $G = D \cap D_\infty$, where D_∞ is the component of the complement of the continuum E that contains the point at infinity. If a sequence of rational functions $r_n(z)$ of the form (1) converges on E to a function $f_1(z)$ and in the domain G to an analytic function $f_2(z)$, and moreover*

$$\overline{\lim}_{n\to\infty}\left[\max_{z\in E}|f_1(z)-r_n(z)|\right]^{1/n}=q<1,$$

then the values of the function $f_1(z)$, $z \in E$, uniquely determine the values of the function $f_2(z)$ in the domain G ($f_2(z)$ is a quasianalytic continuation of $f_1(z)$); in particular, if $f_1(z) = 0$, $z \in E$, then also $f_2(z) = 0$, $z \in G$.

Remark 1. The function $f_2(z)$ in Corollary 1 is not necessarily an analytic continuation of the function $f_1(z)$, since the set of limit points of the sequence of poles of the rational functions $r_n(z)$ may consist of the continuum E and a sequence of isolated points whose derived set coincides with E (the domain G is then infinitely connected). It can be shown that if the conditions of the corollary are fulfilled and the function $f_1(z)$ is analytic on the continuum E (and therefore also in some domain containing the continuum E), then the function $f_2(z)$, which is a quasianalytic continuation of the function $f_1(z)$, coincides in some neighborhood of E with the analytic continuation of the function $f_1(z)$.

Remark 2. The last remark can be applied to the following particular problem: let D be an arbitrary bounded Jordan domain; Γ the boundary of the domain D; G a simply connected domain containing the closed domain $\overline{D}$ in its interior; and let the function $f(z)$ be analytic in the domain G. Wolff ([4]) showed (see also ([3])) that the function $f(z)$ in the domain D can be represented in the form

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - z_n}, \qquad \sum_{n=1}^{\infty} |A_n| < \infty, \qquad z \in D, \tag{2}$$

where $\{z_n\}$ is a sequence of points lying in $G \setminus \overline{D}$, and all limit points of this sequence belong to Γ (so that $\{z_n\}' = \Gamma$).

Thus, the series $\sum \frac{A_n}{z-z_n}$ may converge in the domain D to a function $f(z)$ analytically continuable to the larger domain $G \supset \overline{D}$, while $\sum \frac{A_n}{z-z_n}$ has an infinite set of poles. From the assertion given in Remark 1 it follows that, when the coefficients A_n tend to zero sufficiently rapidly (at the rate of a geometric progression), one may assert that Γ is a cut for the function

$$f(z) = \sum \frac{A_n}{z - z_n}, \qquad z \in D.$$

More precisely, the following is true:

Corollary 2. Let an analytic function $f(z)$ in a simply connected domain D be represented in the form

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - z_n}, \qquad z \in D,$$

where $\{z_n\}$ is a sequence of points lying outside $\overline{D}$ and such that $\{z_n\}' = \Gamma = \overline{D} \setminus D$.

If

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} < 1, \tag{3}$$

then the domain D is the full domain of analyticity of the function $f(z)$.

In other words, in Wolff' s theorem on the representability of a function $f(z)$, analytic in the closed domain $\overline{D}$, by means of functions of the form (2), the coefficients A_n tend to zero sufficiently slowly (in any case,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} = 1$$

).

It can be shown that condition (3) in Corollary 2 cannot be substantially weakened (cf. ([5])).

Theorem 2. Let G_1 and G_2 be arbitrary domains whose boundaries have at least one point of contact; let a sequence of rational functions $r_n(z)$ of the form (1) converge to an analytic function $f_i(z)$ for $z \in G_i,\ i = 1, 2$, and suppose that

$$\overline{\lim_{n\to\infty}} \left[\max_{z\in F} |f_1(z) - r_n(z)|\right]^{1/n} \leq q < 1 \tag{4}$$

for every closed set F belonging to the domain G_1. Then the values of the function $f_1(z)$, $z \in G_1$, uniquely determine the values of the function $f_2(z)$, $z \in G_2$ ($f_2(z)$ is a quasianalytic continuation of $f_1(z)$); in particular, if $f_1(z) = 0$, $z \in G_1$, then also $f_2(z) = 0$, $z \in G_2$.

Condition (4) cannot be weakened for any order of contact of the boundaries of the domains G_1 and G_2; indeed, consider the sequence of rational functions

$$r_n(z) = \frac{z^n}{z^n - 1},$$

and let $G_1 :\ |z| < 1,\ G_2 :\ |z| > 1$. We have

$$\lim_{n\to\infty} r_n(z) = \begin{cases} 0, & z \in G_1, \\ 1, & z \in G_2. \end{cases}$$

At the same time, for any function $\mu(n)$ tending to infinity arbitrarily slowly as $n \to \infty$, we have

$$\overline{\lim_{n\to\infty}} \left[\max_{z\in F} |r_n(z)|\right]^{\mu(n)/n} = 0$$

for every closed set $F \subset G_1$. Note that in this case the domains G_1 and G_2 have the common boundary $|z| = 1$, i.e. the order of contact of the boundaries is, in a certain sense, maximal.

Received
21 VI 1961

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Note: Figure translations are in progress. See original paper for figures.