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Abstract

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HOMOTOPIC DUALITY FOR A SPACE WITH A GROUP OF OPERATORS

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The purpose of the present note is to generalize Spanier–Whitehead duality [1,2] to spaces with a group of operators.

Let G be a topological group. We shall call a topological space X a G-space if the group G acts in X without fixed points, giving rise to a principal fibration $\pi : X \to X_G$ (by X_G we denote the space of trajectories of the group G acting in X). A G-space X is called a G-polyhedron if the spaces X and X_G are finite polyhedra. The set of homotopy classes of admissible (i.e., commuting with the transformations of the group G) mappings of a G-space X into a G-space Y, relative to admissible homotopies, will be denoted by $[X, Y]$.

The join $X*Y$ of two G-spaces X and Y naturally becomes a G-space (the points of the join $X * Y$ are represented by triples (x, y, t) with the identifications $(x, y, 0) \sim (x, y', 0)$; $(x, y, 1) \sim (x', y, 1)$; the group G acts on $X * Y$ by the formula $g(x, y, t) = (g(x), g(y), t)$; here $x, x' \in X$, $y, y' \in Y$, $g \in G$, $0 \leq t \leq 1$). If $\varphi : X \to X'$, $\psi : Y \to Y'$ are admissible mappings of G-spaces, then their join is naturally defined: the admissible mapping $\varphi * \psi : X * Y \to X' * Y'$.

By the homology groups $H_i(X; A)$ (cohomology groups $H^i(X; A)$) we shall always mean the reduced singular homology (cohomology) groups. We note that the group G acts in the homology and cohomology groups of a G-space.

We fix some G-space R. The join $R * \cdots * R$ of k copies of the G-space R will be denoted by R_k. By an R-mapping of a G-space X into a G-space Y we shall mean an admissible mapping $\varphi : R_k * X \to R_k * Y$, where $k = 0, 1, 2, \ldots$. If φ is an admissible mapping of $R_k * X$ into $R_k * Y$, then there is naturally defined an admissible mapping $r_l(\varphi) : R_{k+l} * X \to R_{k+l} * Y$ as the join of the identity mapping of the space R_l and the mapping φ (here $R_{k+l} * X$ and $R_{k+l} * Y$ are considered respectively as $R_l * (R_k * X)$ and $R_l * (R_k * Y)$). Two R-mappings $\varphi : R_k * X \to R_k * Y$ and $\psi : R_l * X \to R_l * Y$ will be called R-homotopic if, for some $n \geq k, l$, the mappings $r_{n-k}(\varphi)$ and $r_{n-l}(\psi)$ are admissibly homotopic. The set of R-homotopy classes of R-mappings of X into Y will be denoted by $\{X, Y\}$. We note that the sets $[R_k * X, R_k * Y]$, in particular the set $[X, Y]$, are naturally mapped into the set $\{X, Y\}$; the set $\{X, Y\}$ may be regarded as the limit of the direct spectrum of the sets $[R_k * X, R_k * Y]$ with respect to the

mappings
$$[R_k * X, R_k * Y] \to [R_{k+1} * X, R_{k+1} * Y].$$

If $\alpha \in \{X, Y\}$ and $\beta \in \{Y, Z\}$, then, by means of composing the R-mappings defining the elements α and β, a composition of these elements $\beta \cdot \alpha \in \{X, Z\}$ can be defined.

In what follows we shall assume that the fixed G-space R is an r-dimensional G-polyhedron, homotopy equivalent to the r-dimensional sphere.

Theorem 1. Let Y be a G-space, aspherical in dimensions $< n$ (i.e., Y is connected and $\pi_i(Y) = 0$ for $1 \leq i < n$), and let X be a G-polyhedron with trajectory space X_G of dimension $\leq 2n-2$. Then the natural mapping of the set $[X, Y]$ into the set $\{X, Y\}$ is one-to-one.

Corollary. If X and Y are G-polyhedra, then for sufficiently large k the natural mapping of the set $[R_k * X, R_k * Y]$ into the set $\{X, Y\}$ is one-to-one.

Every R-mapping of a G-space X into a G-space Y, by virtue of the relation $H_i(X; A) = H_{i+(r+1)k}(R_k * X; A)$, induces a homomorphism of the group $H_i(X; A)$ into the group $H_i(Y; A)$, commuting with the transformations of the group G (here A is any abelian group).

Let X, Y, Z be G-spaces, f an R-mapping of the G-space $X * Y$ into the G-space Z, and A some field. It is known ([3]) that the tensor product $H_i(X; A) \otimes H_{s-i-1}(Y; A)$ is naturally embedded in $H_s(X * Y, A)$; therefore the mapping f_* gives rise to a homomorphism of the group $H_i(X; A) \otimes H_{s-i-1}(Y; A)$ into the group $H_s(Z; A)$, or, in other words, a pairing of the groups $H_i(X; A)$ and $H_{s-i-1}(Y; A)$ in the group $H_s(Z; A)$. If the space Z is homotopy equivalent to the s-dimensional sphere, then the pairing of the groups $H_i(X; A)$ and $H_{s-i-1}(Y; A)$ in the group $H_s(Z; A) = A$ may be regarded as a scalar product (recall that the groups $H_i(X; A)$ and $H_{s-i-1}(Y; A)$ are endowed with the structure of vector spaces over the field A).

Definition. We shall say that the G-polyhedron Y is n-dual to the G-polyhedron X, and write $Y = D_{nX}$, if there exists such an R-mapping of the G-space $X * Y$ into R_n that the scalar product of the vector spaces $H_i(X; A)$ and $H_{n(r+1)-i-2}(Y; A)$, defined by this R-mapping, is nondegenerate for every field A and every $i \geq 0$.

We note the following properties of the duality operator D_n:

1. If $Y = D_{nX}$, then $X = D_{nY}$ (i.e., $D_n = D_n^{-1}$).
2. $R_k * D_{nX} = D_{n+k}X$; $\quad D_n(R_k * X) = D_{n+k}X$.
3. $D_{nX} * D'_{mX} = D_{m+n}(X * X')$.
4. If $Y = D_{nX}$, then there exists an isomorphism $D_n^* : H_i(X) \to H^{n(r+1)-i-2}(Y)$, commuting with the transformations of the group G.
5. If $Y = D_{nX}$, then the polyhedron Y is weakly $(nr + n - 1)$-dual to the polyhedron X in the sense of Spanier–Whitehead ([1]).

6. The space D_{nX} is determined up to R-homotopy type (in other words, if $Y = D_{nX}$ and $Y' = D_{nX}$, then for some $k \geq 0$ there is an admissible homotopy equivalence $\varphi : R_k * Y \to R_k * Y'$).

Let X and Y be G-polyhedra, $Y' = D_{nY}$, and let $u : R_k * Y * Y' \to R_k * R_n = R_{k+n}$ be an R-mapping of $Y * Y'$ into R_n, giving the duality of Y and Y'. Define a mapping P_u of the set $\{X, Y\}$ into the set $\{X * Y', R_n\}$ by means of the following convention. If an element $\alpha \in \{X, Y\}$ is defined by a mapping $\varphi : R_l * X \to R_l * Y$, then the element $P_u(\alpha) \in \{X * Y', R_n\}$ is specified by the superposition ψ of the mappings $\varphi * 1 : (R_l * X) * (R_k * Y') \to (R_l * Y) * (R_k * Y')$ and $1 * u : R_l * (R_k * Y * Y') \to R_l * (R_k * R_n)$ (by 1 we denote the identity mapping; one may speak of the superposition of the mappings $\varphi * 1$ and $1 * u$, since $(R_l * Y) * (R_k * Y') = R_l * (R_k * Y * Y')$; the mapping $\psi : (R_l * X) * (R_k * Y') \to R_l * (R_k * R_n)$, by virtue of the relations $(R_k * X) * (R_k * Y') = R_{k+l} * (X * Y')$, $R_l * (R_k * R_n) = R_{k+l} * R_n$, may be regarded as an R-mapping of $X * Y'$ into R_n).

Main Lemma. The mapping P_u is a one-to-one correspondence between the sets $\{X, Y\}$ and $\{X * Y', R_n\}$.

Definition 2. Let X and Y be G-polyhedra, $X' = D_{nX}$, $Y' = D_{nY}$ their n-dual G-polyhedra; u and v be R-mappings

$X * X'$ in R_n and $Y * Y'$ in R_n, generating these duality relations. We shall say that an element $a \in \{X, Y\}$ is n-dual to an element $\beta \in \{Y', X'\} = \{D_{nY}, D_{nX}\}$, and write $\beta = D_n(u, v)(a)$ (or simply $\beta = D_n(a)$), if $P_u(a) = P_v(\beta)$ (the element $P_u(a)$ lies in the set $\{X * Y', R_n\}$, the element β in the set $\{Y' * X, R_n\}$, but $X * Y' = Y' * X$, and therefore the equality $P_u(a) = P_v(\beta)$ makes sense).

From the fundamental lemma the following follows.

Theorem 2. The mapping $D_n(u, v)$ is a one-to-one correspondence between the sets $\{X, Y\}$ and $\{D_{nY}, D_{nX}\}$.

Let us indicate some properties of the mapping

$$D_n : \{X, Y\} \to \{D_{nY}, D_{nX}\} :$$

1. The mapping D_n takes the composition $\beta \cdot a$ of elements $a \in \{X, Y\}$, $\beta \in \{Y, Z\}$ to the composition $D_n(a) \cdot D_n(\beta)$ of the elements $D_n(\beta) \in \{D_{nZ}, D_{nY}\}$, $D_n(a) \in \{D_{nY}, D_{nX}\}$.
2. The homomorphisms
$$H_i(X; A) \to H_i(Y; A)$$
and
$$H_{n(r+1)-i-2}(D_{nY}; A) \to H_{n(r+1)-i-2}(D_{nX}; A),$$
induced by the elements $a \in \{X, Y\}$ and $D_n(a) \in \{D_{nY}; D_{nX}\}$, are, for any field, adjoint homomorphisms with respect to the scalar products generated by the mappings defining the duality.

The previously indicated properties 1–5 of the duality operator $D_n : X \to D_{nX}$ carry over (with the corresponding changes in the formulations) to the operator

$$D_n : \{X, Y\} \to \{D_{nY}, D_{nX}\}.$$

The description of the operator D_n given above can be simplified in the case when $R = S^r$ and the transformations of the group G on $R = S^r$ are orthogonal transformations. Then the space R_n can be identified with the sphere S^{nr+n-1}, and the transformations of the group G will also be orthogonal transformations on $R_n = S^{nr+n-1}$.

Theorem 3. Let X, Y be disjoint subsets of the sphere $R_n = S^{nr+n-1}$, invariant with respect to the group G. Suppose that X, Y are G-polyhedra with respect to the action of the group G defined in them; Y is a deformation retract of the set $R_n \setminus X$, and Y contains no points diametrically opposite to points of the set X. Then the G-polyhedron Y is n-dual to the G-polyhedron X (i.e. $Y = D_{nX}$). If X', Y' are two other subsets of the sphere R_n satisfying the same conditions, and $X \subset X'$, $Y' \subset Y$, then the inclusion mappings $X \to X'$, $Y' \to Y$ generate n-dual elements of the sets $\{X, X'\}$, $\{Y', Y\}$.

Let us indicate how, under the conditions of this theorem, one constructs the duality-generating mapping $u : X * Y \to R_n$. If a point $a \in X * Y$ is determined by the triple (x, y, t) $(x \in X, y \in Y, 0 \le t \le 1)$, then the point $u(a) \in R_n$ is defined as the point that divides the smaller of the arcs of the great circle joining the points x and y, in the ratio $t : (1-t)$.

The construction of duality described in Theorem 3 can be applied, in particular, to the cyclic group Z_n (for it one may take $R = S^1$, and for $n = 2$ also $R = S^0$) and to the groups $S^1 = U(1)$ of complex numbers of modulus 1, and $S^3 = \mathrm{Sp}(1)$ of quaternions of norm 1 (for these groups one may take for R the group space itself, in which the group acts by left translations).

Example. Let $G = R = S^1$, and let $X = Y_{n,2}$ be the Stiefel manifold of orthonormal 2-frames, in which an element $e^{i\varphi} \in S^1$ acts by rotating each 2-frame through the angle φ in the plane of this frame. The S^1-space Y, n-dual to the S^1-space X, can be constructed as follows. The points of the space Y are pairs (a, b), where $a \in S^{n-1}$, $b \in S^1$, with the identifications $(a, b) \sim (-a, -b)$; the group S^1 acts in Y by the formula $g(a, b) = (a, gb)$.

Remark 1. The G-space R, with the aid of which the duality was constructed, was assumed here to be an r-dimensional polyhedron homotopy equivalent to the r-dimensional sphere. However, at the cost of some complication of the definitions, formulations, and proofs, almost all of the above-listed ⋯

assertions can be carried over to the case where R is an arbitrary G-space homotopy equivalent to a sphere.

Remark 2. The above-indicated duality of G-spaces is closely connected with the duality of fiber spaces, the construction of which will be the subject of a separate note [4].

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Note: Figure translations are in progress. See original paper for figures.