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## Abstract

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## MATHEMATICS

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# ON THE EMBEDDING OF SIMPLY CONNECTED MANIFOLDS IN EUCLIDEAN SPACE

*(Presented by Academician P. S. Aleksandrov on 23 I 1961)*

We shall consider smooth closed manifolds and their smooth mappings. As usual, we shall call a mapping $f : M^n \to W^m$ **regular** if its Jacobian matrix has rank $n$ at every point, and **completely regular** if the preimage of any point $f^{-1}(w)$, where $w \in W^m$, contains no more than two points. Our aim is to investigate the possibility of a smooth embedding $M^n \subset E^{2n-1}$. The method generalizes the method of Whitney' s work (2), devoted to embeddings $M^n \subset E^{2n}$, on the basis of ideas of L. S. Pontryagin concerning the homotopy groups of spheres (1).

**Theorem 1.** *Every simply connected odd-dimensional manifold* $M^n$ *with* $n > 6$ *can be smoothly embedded in Euclidean space* $E^{2n-1}$.

The proof of this theorem is based on a series of lemmas devoted to the study of regular mappings as a whole. With the help of well-known techniques (developed by Whitney) it is easily proved:

**Lemma 1*.** *For every regular mapping* $f : M^n \to E^{2n-k}$ *with* $k < \left[\frac{n}{2}\right]$, *there exists a* $C^1$*-close regular mapping* $g : M^n \to E^{2n-k}$ *such that:*

1) *the equation* $g(x) = g(y)$ *defines a closed submanifold* $\widetilde{M}_g^k \subset M^n \times M^n \setminus \Delta(M^n)$, *where* $\Delta$ *is the diagonal mapping;*
2) *the projection* $p : M^n \times M^n \to M^n$, *considered on the submanifold* $\widetilde{M}_g^k$, *is a smooth homeomorphism;*
3) *the mapping* $g$ *is completely regular; on the special submanifold* $M_g^k = p(\widetilde{M}_g^k) \subset M^n$ *it is a two-sheeted covering.*

It follows from the lemma that the special manifold $M_g^k$ decomposes into some number $s$ of special pairs of mutually homeomorphic connected components

$$\bigcup_{i=1}^{s}\left(M_{g,1}^{k,i}\cup M_{g,2}^{k,i}\right)$$

such that $g\left(M_{g,1}^{k,i}\right)=g\left(M_{g,2}^{k,i}\right)$, and some number $t$ of connected manifolds

$$\bigcup_{j=1}^{t} M_g^{k,j},$$

on which the mapping $g$ is a nontrivial 2-covering. Thus,

$$M_g^k=\left(\bigcup_{j=1}^{t} M_g^{k,j}\right)\cup\left(\bigcup_{i=1}^{t}\left(M_{g,1}^{k,i}\cup M_{g,2}^{k,i}\right)\right).$$

**Definition.** We shall call a manifold $M^n$ **$k$-parallelizable** if the $\varepsilon$-neighborhood $U_\varepsilon^{(k)}$ of the $k$-dimensional skeleton of a smooth triangulation of the manifold $M^n$ is parallelizable for sufficiently small $\varepsilon$.

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* For $k=1$ this lemma is contained in Whitney' s works; see, for example, ($^3$).

Obviously, for $n>2k+2$ our definition does not depend on the triangulation, and a $k$-connected manifold is $k$-parallelizable. For $k=1$ this definition means simply orientability. We also note that a $k$-parallelizable manifold is $(k-1)$-parallelizable. In what follows we assume that $n\geqslant 2k+3$.

**Lemma 2.** *If the manifold $M^n$ is $k$-parallelizable, then the singular submanifold $M_g^k\subset M^n$ has a trivial normal bundle in the manifold $M^n$ and is a $\pi$-manifold (i.e., the normal bundle under the embedding $M_g^k\subset E^m$ is trivial for $m\geqslant 2k+3$).*

The proof of this lemma is based on the fact that the manifold $M^n\times M^n\setminus\Delta(M^n)$ is also $k$-parallelizable and that the normal bundle of a manifold of dimension at most $k$ in a $k$-parallelizable manifold of large dimension is arranged in the same way as in a Euclidean space of the same dimension.

Let $n$ be even, $k=1$, and let the manifold $M^n$ be orientable.

**Lemma 3.** *The singular submanifold $M_g^1\subset M^n$ consists only of singular pairs of circles.*

Suppose, to the contrary, that the manifold $M_g^1$ contains a circle $S_g^1\subset M_g^1$ on which the mapping $g$ is a connected 2-fold covering. Obviously, $g(S_g^1)=S^1\subset E^{2n-1}$. Choose a system $(W_1,\dots,W_{n-1})$ of independent vector fields, transversal to $S_g^1\subset M^n$ and tangent to the manifold $M^n$.

Then, roughly speaking, there is defined a decomposition of the normal bundle of $g(S_g^1)\subset E^{2n-1}$ into the sum of plane bundles $\mu_i^{(2)}$, $i=1,\dots,n-1$, generated by the vectors $W_i$. Each plane bundle $\mu_i^{(2)}$ is transversal to the circle $g(S_g^1)$ and has the monodromy matrix $A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e., each bundle $\mu_i^{(2)}$ is nonorientable. The Whitney sum of an odd number of such bundles is also nonorientable and, consequently, nontrivial. Since $n-1$ is odd, whereas the normal bundle to a circle in Euclidean space must be trivial, we arrive at a contradiction. The lemma is proved.

For the general case, when $k = 1$, we have:

**Lemma 4.** *The number $t(g)$ of connected singular coverings is always even. The mapping $g$ is regularly homotopic to a mapping $g_1$ that has no singular nontrivial 2-fold coverings.*

The proof of this lemma is of a somewhat different nature and is based on the study of the projection

$$M^n \xrightarrow{\tilde{g}} E^{2n} \xrightarrow{\pi} E^{2n-1},$$

where $g = \pi \circ \tilde{g}$, and the mapping $\tilde{g}$ is completely regular (one can always easily reduce it to this form by a small $C^1$-perturbation of the mapping in the space $E^{2n}$, projecting a small perturbation into $E^{2n-1}$, which, obviously, preserves the properties of Lemma 1).

The behavior of the projection is described more fully by the following trivial lemmas:

**Lemma 4a.** *If the mapping $\tilde{g} : M^n \to E^{2n}$ and $\pi\tilde{g} : M^n \to E^{2n-1}$ are regular, then the mapping $\tilde{g}$ is regularly homotopic to an embedding and has an even number of pairs of singular points.*

**Lemma 4b.** *A connected singular covering of the mapping $g$ can arise under projection only from an odd number of pairs of singular points of the mapping $\tilde{g}$.*

*A singular pair can arise under projection only from an even number of pairs of singular points of the mapping $\tilde{g}$.*

**Lemma 4c.** There exists a regular homotopy $\tilde{g}_t$ of the map $\tilde{g} = \tilde{g}_0$ such that:

1) the maps $\tilde{g}_t$ and $\pi\tilde{g}_t$ are regular for $t \leqslant 1$ and are completely regular for $t = 1$; the map $\pi\tilde{g}_1$ satisfies Lemma 1;
2) under the projection $\pi$, from one pair of singular points there arise double points, and from nothing—singular pairs of circles of the map $\pi\tilde{g}_1$.

In what follows we shall consider only such maps

$$g : M^n \to E^{2n-1}$$

that have no linked singular double points. We shall also assume that $\pi_1(M^n) = 0$. Following Pontryagin (¹), we shall define the invariant of a singular pair and the invariant of a map $g$.

**Definition of the invariant of a singular pair.** Let $S_1^1$ and $S_2^1 \subset M^n$, $g(S_1^1) = g(S_2^1)$. Consider a pair of disks $\sigma_1^2, \sigma_2^2 \subset M^n$ such that $\sigma_1^2 \cap \sigma_2^2 = \emptyset$ and $\partial\sigma_1^2 = S_1^1$, $\partial\sigma_2^2 = S_2^1$. Choose a system of vector fields $W_j^{(i)}$, $i = 1, 2$; $j = 1, \dots, n-2$, tangent to $M^n$ and orthogonal to $\sigma_i^2$. Put $W_{n-1}^{(i)} = \partial\sigma_i^2/\partial t$, where $t$ are the radii of the films (i.e., transverse to $S_i^1$ and to $W_j^{(i)}$, $j \leqslant n-2$). Obviously, the vectors $g(W_j^{(i)}) = V_{j+(i-1)i}$, transverse to $g(S_i^1)$, are defined and independent. They determine an element

$$\alpha \in \pi_1(GL(2n-2)) = Z_2.$$

**Lemma 5.** If the generating element of the group $H^n(M^n, Z_2)$ has the form $Sq^2(x)$, $x \in H^{n-2}(M^n, Z_2)$, then the disks $\sigma_i^2$ and the fields $W_j^{(i)}$ can be chosen so that $\alpha = 0$.

In the case when

$$H^n(M^n, Z_2)/\operatorname{Im} Sq^2 = Z_2,$$

the invariant $\alpha$ of a singular pair does not depend on the choice of the disks $\sigma_i^2$. In this case we shall regard the sum of the invariants

$$\sum_k \alpha_k$$

over all singular pairs $S_k = (S_{g,1}^{1,k} \cup S_{g,2}^{1,k})$ as the invariant of the map

$$g : M^n \to E^{2n-1},$$

if it has no linked singular double points.

**Lemma 6.** For a simply connected odd-dimensional manifold $M^n$, $n = 4l + 3$, the invariant

$$\sum_k \alpha_k = 0$$

for any regular map

$$g : M^n \to E^{2n-1},$$

possessing all the properties of Lemma 1*.

This lemma is an important step, and its direct geometric proof is quite difficult. But from the recent works of Hirsch (⁴), devoted to regular maps, it is extracted more simply.

Let $S_1$ and $S_2$ be two singular pairs of the map

$$g : M^n \to E^{2n-1}$$

such that $\alpha(S_1) = \alpha(S_2)$.

**Lemma 7.** There exists a regular homotopy $g_t$ of the map $g = g_0$ such that the map $g_1$ satisfies Lemma 1 and has two fewer singular pairs than the map $g = g_0$.

The proof generalizes the well-known proof of Whitney ($^2$) for pairs of singular points. We glue into the manifold $M^n$ the rings

$$B_1 = S_1^1 \times I \quad \text{and} \quad B_2 = S_2^1 \times I$$

so that $S_i^1 \times \varepsilon$ form the pair $S_1$, and $S_i^1 \times (1-\varepsilon)$ form the pair $S_2$. It is necessary that $B_1 \cap B_2 = \emptyset$. On the rings we prescribe vector fields $W_j^{(i)}$, $i = 1, 2;\ j = 1, \dots, n-2$, extended from the disks $\sigma_i^2$, determining the invariants $\alpha(S_i)$. One can easily arrange that the frames

$$(\tau_i, g(W_j^{(1)}), g(W_j^{(2)})),$$

where $\tau_i$ are vector fields tangent to $g(S_i)$, determine opposite orientations for $i = 1, 2$. Next we glue in a "Whitney cell"

$$\psi : \sigma^2 \times S^1 \to E^{2n-1}$$

such that

$$\psi(\sigma^2 \times S^1) \cap g(M^n) = g(B_1) \cup g(B_2).$$

It is also necessary that on the boundaries the mapping $\psi$ has certain compatibility properties. After this, by virtue of the coincidence of the invariants of the pairs $S_1$ and $S_2$, one can specify, in a small neighborhood $U(\psi(\sigma^2 \times S^1))$, a suitable coordinate system, one of whose coordinates is a point of the circle, two others are a standard 2-frame on $\sigma^2$, and the remaining ones satisfy our boundary conditions and are independent of these. Having specified the coordinates, we perform in the neighborhood $U(\psi(\sigma^2 \times S^1))$ a Whitney deformation with the coordinate of the circle held fixed.

Iterating this construction and using Lemma 6, we arrive at a mapping

$$g_S : M^n \to E^{2n-1},$$

which has special pairs only with zero invariant.

After this, with the mapping $g_S$ one can proceed in two ways: either, following (2), glue in additional pairs with zero invariants and apply Lemma 7, or else

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* If $n = 4l + 1$, one can assert the existence of maps $g : M^n \to E^{2n-1}$ with zero invariant, since there exists an immersion $M^n \to E^{2n-2}$ ($^4$).

carry out the direct separation of a pair with zero invariant.* In either case we arrive at an embedding. Thus, the theorem follows from the preceding lemmas.

I note that the lemmas imply the following conditional

**Theorem 2.** *Let $n = 2l$, $n \geqslant 6$, and $\pi_1(M^n) = 0$. An embedding $M^n \subset E^{2n-1}$ exists if and only if there exists an immersion $M^n \to E^{2n-2}$ (see (4)).**

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* The method of separating one special pair with zero invariant was suggested to me by D. B. Fuks, who kindly read the present work.

** A more precise investigation shows that, for $n = 4l + 2$, the invariant $\alpha$ defined above is a homotopy invariant of the manifold; it does not depend on the immersion

$$g : M^n \to E^{2n-1}$$

when $n \not\equiv 1 \pmod 4$.

*Note: Figure translations are in progress. See original paper for figures.*