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Abstract

Full Text

ON INFINITE-DIMENSIONAL HOMOGENEOUS SPACES

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The purpose of this note is to show how various homogeneity conditions affect the dimensional properties of infinite-dimensional spaces. It turns out that if a locally bicomact group of transformations (whose space is finally compact) acts transitively on an infinite-dimensional locally bicomact space, then such a space is strongly infinite-dimensional*. Apparently, this result, in some form, remains valid under weaker homogeneity assumptions as well. The following proposition (which, however, has a rather special character) also speaks in favor of this hypothesis:

*If an infinite-dimensional metrizable locally bicomact space X is doubly homogeneous (i.e., for any two points there exists a homeomorphism carrying these points into some fixed pair of points), then it cannot have transfinite dimension ω^{**} .*

Indeed, suppose that X has transfinite dimension ω . Let x be an arbitrary point of the space X , and let U be a neighborhood of this point such that $X[U]$ is bicomact, is not equal to X , and the boundary $\text{Fr}_X U$ of the set U is finite-dimensional. Let $n = \text{ind Fr}_X U$; we shall show that then $\text{ind } X \leq n+1$. In view of the homogeneity of the space X , it is enough to show that every neighborhood Ox of the point x contains a neighborhood V such that $\text{ind Fr}_X V \leq n$. Let $y \in X \setminus X[U]$. For every point $z \in \text{Fr}_X Ox$, consider a homeomorphism f_z of the space X onto itself such that $f_z x = x$, $f_z y = z$. Let f_1, \dots, f_s be such a finite collection of homeomorphisms for which the set

$$\bigcup_{i=1}^s f_i(X \setminus X[U])$$

contains $\text{Fr}_X Ox$. Then, as V , one can obviously take the set

$$\left\{ \bigcap_{i=1}^s f_i U \right\} \cap Ox.$$

We pass to the proof of the proposition mentioned at the beginning of the paper.

Theorem. *Let G be an infinite-dimensional locally bicomact group, H a closed subgroup in G , and let the quotient space $B = G/H$ be infinite-dimensional.*

Then B is strongly infinite-dimensional; moreover, it contains a Hilbert parallelepiped.

Proof. First of all we note that if G_1 is an open subgroup in G and $H_1 = H \cap G_1$, then the quotient space $B_1 = G_1/H_1$ is also infinite-dimensional. Indeed, the natural mapping $f : B_1 \rightarrow B$

* According to the definition proposed by P. S. Aleksandrov (see (3), p. 14), a space is called **strongly infinite-dimensional** if there exists in it a countable system of pairs of closed sets $\{A_i, B_i\}$, $A_i \cap B_i = \Lambda$, such that for every system of closed sets C_i separating A_i and B_i , the set $\bigcap_i C_i$ is nonempty.

** Transfinite dimension is defined by induction: a space has dimension $\leq \alpha$ if each of its points has arbitrarily small neighborhoods with boundaries of dimension $< \alpha$; in particular, a space has transfinite dimension ω if each of its points has arbitrarily small neighborhoods with finite-dimensional boundary.

is an embedding of the set B_1 into an open subset of the space B ; moreover, B turns out to be the union of pairwise disjoint locally homeomorphic open sets, namely the sets that are images, under the canonical projection of G onto B , of the double adjacent classes of the group G with respect to the subgroups H and G_1 .

As is known (see (2), p. 39), every locally bicomact group contains an open projective-Lie subgroup*; this subgroup, in turn, contains an open subgroup of bicomact origin**, which is also projective-Lie ((2), p. 5). Therefore, in view of the remark made above, without loss of generality one may assume that the group G itself is projective-Lie and of bicomact origin.

Let τ be the local weight of the topological group G ; let θ be the least ordinal number of cardinality τ . Represent the group G as the limit of a completely ordered inverse spectrum of locally bicomact groups $\{G_\alpha, \pi_\beta^\alpha\}$ of length θ , satisfying the following conditions: 1) for every limit ordinal $\alpha < \theta$ we have $G_\alpha = \lim_{\beta < \alpha} G_\beta$; 2) the kernel of the homomorphism

$$\pi_1 : G \rightarrow G_1$$

is a bicomact normal divisor of the group G (whence it follows that for every $\alpha < \theta$ the kernel of the homomorphism $\pi_\alpha : G \rightarrow G_\alpha$ is bicomact); 3) the group G_1 is a Lie group; for every $\alpha < \theta$ the kernel of the homomorphism $\pi_{\alpha+1}^{\alpha+1} : G_{\alpha+1} \rightarrow G_\alpha$ is a Lie group. Such an inverse spectrum is called a Lie series of the group G . In the book of L. S. Pontryagin (6) a Lie series is constructed for bicomact groups; the construction given there remains fully valid also for projective-Lie locally bicomact groups. It remains only to verify that the limit group G' of the spectrum $\{G_\alpha, \pi_\beta^\alpha\}$ is isomorphic to the group G . The natural homomorphism $\pi : G \rightarrow G'$, generated by the homomorphisms $\pi_\alpha : G \rightarrow G_\alpha$, is, by the construction of the spectrum, a monomorphism. We shall show that it is an epimorphism. Let $g' = \{g_\alpha\}$, $g_\alpha \in G_\alpha$, be an arbitrary element of the

group G' . In accordance with condition 2), the sets $\pi_\alpha^{-1}g_\alpha$ are bicomact and, moreover, if $\alpha < \beta$, then $\pi_\alpha^{-1}g_\alpha \supset \pi_\beta^{-1}g_\beta$; hence

$$\bigcap_{\alpha < \theta} \pi_\alpha^{-1}g_\alpha \neq \Lambda.$$

If $g \in \bigcap_{\alpha < \theta} \pi_\alpha^{-1}g_\alpha$, then $\pi_\alpha g = g_\alpha$, i.e. $\pi g = g'$. Thus $\pi G = G'$. From condition 2) it follows that the group G' is locally bicomact***), and therefore, in accordance with Theorem 12 of (6), the homomorphism π is a topological isomorphism.

We now proceed to the construction of the spectrum for the quotient space B . From condition 2) it follows that the homomorphisms π_α are closed mappings, and therefore for each α the subgroup $H_\alpha = \pi_\alpha H$ is closed in G_α ; moreover

$$H = \lim_{\alpha < \theta} H_\alpha.$$

Let $B_\alpha = G_\alpha/H_\alpha$. The spaces B_α also, obviously, form an inverse spectrum. We shall denote by φ_β^α the mapping $B_\alpha \rightarrow B_\beta$ induced by the homomorphism π_β^α . By one theorem of Mostert ((4), theorem 4),

$$B = \lim_{\alpha < \theta} B_\alpha.$$

We shall show that for each $\alpha < \theta$ the mapping $\varphi_\alpha^{\alpha+1}$ is a locally trivial fibration whose fiber is a manifold. Let

$$N_\alpha = \text{Ker } \pi_\alpha^{\alpha+1}.$$

In accordance with conditions 2) and 3), N_α is a compact Lie group. The group N_α acts on the space $B_{\alpha+1}$, and the fibration

$$\varphi_\alpha^{\alpha+1} : B_{\alpha+1} \rightarrow B_\alpha$$

is a fibration onto the orbits of the group N_α . These orbits are homeomorphic to the quotient space

$$M_\alpha = N_\alpha/N_\alpha \cap H_{\alpha+1},$$

which is

* A locally bicomact group is called **projective-Lie** if every neighborhood of its identity contains a normal divisor such that the quotient group by it is a Lie group.

** A topological group is called a **group of bicomact origin** if it is generated by some bicomact neighborhood of the identity.

***) Since G_1 is locally bicomact, and the mapping π_1 is closed and bicomact ((5), theorem 4).

manifold. The local triviality of the fibration $\varphi_\alpha^{\alpha+1}$ follows from Gleason's theorem ((¹), Theorem 3.6), in which the local triviality of the fibration onto the orbits is established under the assumption that the stationary subgroups of different points are conjugate to one another. In our case this condition is evidently satisfied, since the stationary subgroup of the element $aH_{\alpha+1} \in B_{\alpha+1}$ ($a \in G_{\alpha+1}$) is $aH_{\alpha+1}a^{-1} \cap N_\alpha$.

Thus, let us prove that the space B contains a Hilbert parallelepiped. To this end we prove by induction that each B_α contains a certain cube I_α in such a way that the mapping $\varphi_\alpha^{\alpha+1}$ on the cube $I_{\alpha+1}$ coincides with the projection of $I_{\alpha+1}$ onto I_α parallel to a finite-dimensional face of this cube whose dimension is equal to the dimension of the manifold M_α . As I_1 we choose a closed neighborhood in the manifold B_1 homeomorphic to a cube. Suppose that, for all $\alpha < \beta$, the cubes I_α have already been constructed. Then, if β is not a limit transfinite number, then, since the set $J_{\beta-1}$ is contractible,

$$(\varphi_{\beta-1}^\beta)^{-1}I_{\beta-1} = I_{\beta-1} \times M_{\beta-1},$$

and as I_β we take the cube $I_{\beta+1} \times I'$, where I' is some closed neighborhood in the manifold $M_{\beta-1}$ homeomorphic to a cube. If, however, β is a limit number, then we set

$$I_\beta = \lim_{\alpha < \beta} I_\alpha.$$

The cube I_β is canonically embedded in B_β , since, by the Moestert theorem cited above and condition 1),

$$B_\beta = \lim_{\alpha < \beta} B_\alpha.$$

Then

$$I = \lim_{\alpha < \theta} I_\alpha$$

will be the desired parallelepiped in the space B . The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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