ON THE GEOMETRY OF CONES IN A BANACH SPACE

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Abstract

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ON THE GEOMETRY OF CONES IN A BANACH SPACE

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1. According to M. G. Kreĭn ([1]), a cone K in a real Banach space E is called normal if there exists such a $\delta > 0$ that from $e_1, e_2 \in K$, $\|e_1\| = \|e_2\| = 1$, it follows that

$$\|e_1 + e_2\| \geq \delta. \tag{1}$$

I. A. Bakhtin showed ([4,5]) that this definition is equivalent to the following: a cone K is called normal if there exists such a constant $M > 0$ that from the inequalities $\theta \leq x \leq y$ there follows the inequality $\|x\| \leq M\|y\|$ (here, as usual, the sign $\leq$ denotes the relation of partial ordering generated in E by the cone K).

Starting from I. A. Bakhtin's criterion for normality of a cone, in ([3]) the following generalization of this concept was proposed for the case of Banach spaces with two cones: let K_0 and K be cones in the real Banach space E, with $K_0 \subset K$. By the sign $\leq$ we shall denote the relation of partial ordering generated in E by the cone K.

Definition 1. The cone K_0 is called K-normal if there exists such a constant M that for any elements $x, y \in K_0$, from the inequality $x \leq y$ (i.e. $y - x \in K$) it follows that $\|x\| \leq M\|y\|$.

The definition of normality of a cone due to M. G. Kreĭn is naturally generalized to the case of spaces with two cones in the following way:

Definition 2. We shall call the cone K_0 K-normal ($K_0 \subset K$) if there exists such a positive δ that from $f_1^0 \in K_0$, $f_2 \in K$, $f_1^0 + f_2 \in K_0$, there follows the inequality

$$\|f_1^0 + f_2\| \geq \delta \max\{\|f_1^0\|, \|f_2\|\}. \tag{2}$$

Theorem 1. *Definitions of K-normality 1 and 2 are equivalent.*

Let us prove, for example, that if the cone K_0 is K-normal in the sense of Definition 1, then it has the same property also in the sense of Definition 2.

Under the contrary supposition, there would be found two such sequences $f_n^0 \in K_0$, $f_n \in K$, that $f_n^0 + f_n \in K$ and

$$\|f_n^0 + f_n\| < \frac{1}{n^3} \max\{\|f_n^0\|, \|f_n\|\} \quad (n = 1, 2, ...). \tag{3}$$

From inequalities (3) it follows that, for all $n = 2, 3, ...$, the inequalities $\frac{1}{2}\|f_n^0\| \leq \|f_n\| \leq 2\|f_n^0\|$ are valid, whence $\|f_n^0 + f_n\| < \frac{1}{n^3} 2\|f_n^0\|$ $(n = 2, 3, ...)$. The last inequality means that the series

$$\sum_{n=2}^{\infty} \left(n \frac{f_n^0}{\|f_n^0\|} + n \frac{f_n}{\|f_n^0\|} \right) \tag{4}$$

converges strongly. Denote the sum of this series by f^*. Obviously, $f^* \in K_0$ and $nf_n^0/\|f_n^0\| \leq f^*$ $(n = 2, 3, ...)$. Since $nf_n^0/\|f_n^0\| \in K_0$ $(n = 2, 3, ...)$, it follows from the last inequality and the K-normality of the cone K_0, in the sense of Definition 1, that $nf_n^0/\|f_n^0\| = n \leq M\|f^*\|$ $(n = 2, 3, ...)$. We have arrived at a contradiction.

Let us note that the K-normality of the cone K_0 does not imply the normality of the cone K. Thus, for example, every locally compact [(2)] cone K_0 will be K-normal with respect to any cone K $(K \supset K_0)$. Another nontrivial example of a K-normal cone in the space $C^1[0, 1]$, with norm

$$\|x(t)\|_{C^1} = \max_{0 \leq t \leq 1} |x(t)| + \max_{0 \leq t \leq 1} |x'(t)|,$$

for the cone K of all nonnegative functions, is formed by the cone K_0 of all nonnegative convex functions that vanish at the endpoints of the segment $[0, 1]$. The cone K does not possess the normality property. However, the cone K_0 is a K-normal cone: from $x(t) \leq y(t)$ $(x(t), y(t) \in K_0)$ it follows that $\|x(t)\|_{C^1} \leq \|y(t)\|_{C^1}$. Let us note that, moreover, the cone K_0 under consideration will be K-regular (but not fully K-regular).

2. Let E be a real Banach space, and let K_0 and K be cones in it; let u_0 be some fixed nonzero element of K_0. Everywhere below the sign $\overset{(0)}{\leq}$ will denote the semi-ordering generated in E by the cone K_0, while the sign $\leq$ will denote the semi-ordering established in E by the cone K. By E_{u_0} we denote the set of those elements $x \in E$ for which, for some nonnegative a and b, the inequalities

$$-bu_0 \overset{(0)}{\leq} x \leq au_0. \tag{5}$$

are satisfied. Denote by $b(x)$ and $a(x)$ the exact lower bounds of the numbers b and a satisfying (5). The larger of the numbers $b(x)$, $a(x)$ we shall denote by $\|x\|_{u_0}$. In view of (5) and the closedness of the cone,

$$-\|x\|_{u_0} u_0 \overset{(0)}{\leq} x \leq \|x\|_{u_0} u_0,$$

where $\|x\|_{u_0}$ is the smallest of the numbers $c > 0$ satisfying the inequalities

$$-cu_0 \overset{(0)}{\leq} x \leq cu_0.$$

The functional $\|\cdot\|_{u_0}$, defined on the elements of the set E_{u_0}, satisfies the following two axioms of a norm: 1) $\|x\|_{u_0} \geq 0$ $(x \in E_{u_0})$, and $\|x\|_{u_0} = 0$ if and only if $x = \theta$; 2) if $x, y \in E_{u_0}$, then $\|x + y\|_{u_0} \leq \|x\|_{u_0} + \|y\|_{u_0}$.*

As a generalization of a theorem of M. A. Krasnosel' skii (2), one can prove the following theorem.

Theorem 2. *In order that the cone K_0 be K-normal, it is necessary and sufficient that the inequality*

$$\|x\|_E \leq M\|x\|_{u_0}\|u_0\|_E \qquad (x \in E_{u_0},\ u_0 \in K_0), \tag{6}$$

hold, where the constant M depends neither on $u_0 \in K_0$ $(u_0 \neq \theta)$ nor on $x \in E_{u_0}$.

Proof of necessity. Suppose that such a constant M does not exist–suppose that there are sequences x_n, u_n such that $u_n \in K_0$ $(u_n \neq \theta)$, $x_n \in E_{u_n}$ $(n = 1, 2, ...)$, and

$$\|x_n\|_E \geq n\|x_n\|_{u_n}\|u_n\|_E \qquad (n = 1, 2, ...).$$

Then

$$-u_n/n\|u_n\|_E \overset{(0)}{\leq} x_n/\|x_n\|_E \leq u_n/n\|u_n\|_E \qquad (n = 1, 2, ...).$$

Therefore, if

$$y_n = u_n/n\|u_n\|_E - x_n/\|x_n\|_E, \qquad z_n^0 = u_n/n\|u_n\|_E + x_n/\|x_n\|_E,$$

then $z_n^0 \in K_0$, $z_n^0 + y_n \in K_0$, and $y_n \in K$ $(n = 1, 2, ...)$. But then, by virtue of the K-normality of the cone K_0, the inequality

$$\|z_n^0 + y_n\| \geq \delta \max\{\|z_n^0\|, \|y_n\|\} \geq \delta(1 - 1/n)$$

must hold, and therefore for $n \geq 2$, $\|z_n^0 + y_n\| \geq \delta/2$. On the other hand,

* In the case when $K_0 = K$, the functional $\|x\|_{u_0}$ coincides with the so-called u_0-norm; the u_0-norm has been used by many authors.

$z_n^0 + y_n\| = 2/n \to 0$. The contradiction obtained proves the necessity of inequality (6) for the K-normality of the cone K_0.

We shall also prove the **sufficiency** by contradiction: suppose inequality (6) is satisfied, but the cone K_0 does not possess the property of K-normality. Then there exists a sequence $f_n^0 \in K_0$ and an element $f^* \in K_0$ such that

$$nf_n^0/\|f_n^0\| \leq f^* \quad (n = 2, 3, ...)$$

(see the proof of Theorem 1). Therefore

$$\|nf_n^0/\|f_n^0\|\|_{f^*} \leq 1 \quad (n = 2, 3, ...),$$

and hence, by virtue of (6),

$$n = \|nf_n^0/\|f_n^0\| \leq M\|f^*\| \quad (n = 2, 3, ...),$$

which is impossible. The theorem is proved.

3. We shall say that the cone K_0 is K-generating ($K_0 \subset K$) if, for every $x \in E$, one can indicate an element $u \in K_0$ such that $x \leq u$. In other words, the cone K_0 is called K-generating if every element $x \in E$ is representable in the form $x = u - v$, where $u \in K_0$, $v \in K$. The concept of a K-generating cone is a natural generalization of the concept of a generating cone ([1]). Examples of K-generating (but not generating) cones K_0 may be, for example, the cone K_0 of nonnegative nonincreasing functions in the space $C[0, 1]$ with K the cone of nonnegative functions, and the cone K_0 of convex nonnegative functions in the space $C_0^1[0, 1]$ of continuously differentiable functions vanishing at the endpoints of $[0, 1]$, with K the cone of nonnegative functions.

M. G. Krein showed ([1]) that the cone K is normal if and only if the semigroup K^* of linear functionals positive on K generates E^*, i.e., for every $f \in E^*$ there is a representation

$$f = g - h,$$

where $g, h \in K^*$.

Theorem 3. *Let the cone K_0 be K-generating, and let K_0^* be a cone. Then K^* is a K_0^*-normal cone.*

We do not know whether the converse assertion is true.

4. In the case where the cone K is generating, for every $x \in E$ there exists an element $u(x) \in K$ such that $x \leq u(x)$ and $\|u(x)\| \leq M\|x\|$, where the constant M does not depend on the choice of the element x. M. A. Krasnosel' skii proposed calling this property the property of **nonflattening** of the cone. About 20 years ago, the nonflattening of every generating cone was proved by M. G. Krein and V. L. Shmul' yan. M. A. Krasnosel' skii showed that nonflattening of a generating cone plays an essential role in the study of derivatives with respect to directions of the cone (it was found that differentiability with respect to directions of the cone implies the existence of ordinary derivatives; conditions for complete continuity of derivatives with respect to the cone were found, etc.). In connection with this, I. A. Bakhtin reconsidered the question of nonflattening of generating cones and proposed a new proof of the nonflattening of every generating cone.

We shall say that the cone K_0 is called K-nonflattening if, for every $x \in E$, there exists an element $u(x) \in K_0$ such that $x \leq u(x)$ and $\|u(x)\| \leq M\|x\|$, where M is a constant.

Theorem 4. *The classes of K-nonflattening and K-generating cones coincide.*

Not being able to dwell on the proof of this theorem, we note that in its proof we used constructions of I. A. Bakhtin.

5. In connection with the fact that, in applications of the theory of spaces with a cone to nonlinear problems, an important role is played by the possibility of passing to the limit along monotone bounded sequences, M. A. Krasnosel' skii ([2,3]) proposed the following definitions. A cone K_0 is called K-regular if, for any sequence $x_n \in K_0$, from the inequalities

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots; \tag{7}$$

$$x_n \leq u \qquad (u \in K_0,\ n = 1, 2, ...) \tag{8}$$

imply the convergence of the sequence x_n. The cone K_0 is called **fully K-regular** if, for every sequence $x_n \in K_0$, the inequalities (7) and

$$\|x_n\| \leq M \qquad (n = 1, 2, ...) \tag{9}$$

imply its convergence. The relation between these notions in the case $K = K_0$ was considered in [2]; some theorems for the case $K \neq K_0$ are indicated in [3]. Here we shall give one supplement to these theorems.

Theorem 5. *Every fully K-regular cone K_0 is regular.*

From this theorem it follows, in particular, that every fully K-regular cone K_0 is K-normal.

6. Consideration of the weak topology makes it possible to introduce a further generalization of the notion of regularity of a cone. We shall say that the cone K_0 is **weakly K-regular** if, for every sequence $x_n \in K_0$, the inequalities (7) and (8) imply its weak convergence. We shall call the cone K_0 **weakly fully K-regular** if it is K-normal and if the inequalities (7) and (9) ($x_n \in K_0$, $n = 1, 2, ...$) imply the weak convergence of the sequence x_n. We give, without proof, several theorems on weakly K-regular and weakly fully K-regular cones.

Theorem 6. *A weakly K-regular cone K_0 is a K-normal cone.*

Theorem 7. *Every weakly fully K-regular cone K_0 is weakly K-regular.*

The converse theorem is not true.

Theorem 8. *A weakly fully K-regular cone K_0 is fully K-regular if and only if it is K-regular.*

Theorem 9. *In order that the cone K_0 be weakly K-regular or weakly fully K-regular, it is sufficient that the space E be weakly complete and that the linear hull of the set K^* be dense in E^*.*

From the last theorem it follows, in particular, that in weakly complete spaces ($\mathcal{L}_p$, etc.) every normal cone is weakly regular and weakly fully regular.

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Note: Figure translations are in progress. See original paper for figures.