V. GAVRILOV

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Abstract

Full Text

MATHEMATICS

V. GAVRILOV

ON LIMITS ALONG SEQUENCES OF POINTS OF NORMAL MEROMORPHIC FUNCTIONS

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1. Let a meromorphic function $f(z)$ be given in the unit disk $D : |z| < 1$. Following Lehto and Virtanen (1), we shall call this function **normal in the disk** D if, for an arbitrary family $\{T(z)\}$ of one-to-one conformal mappings of the disk D onto itself, the family of functions $\{f(T(|z|))\}$ is normal in the disk D in the sense of Montel. Lehto and Virtanen showed that a normal meromorphic function $f(z)$ will have a definite angular boundary value a at some point $e^{i\theta}$ on the boundary of the disk D, if a is an asymptotic value of the function $f(z)$ along some Jordan curve lying in the closure $\overline{D}$ and ending at $e^{i\theta}$.

An immediate consequence of this result and of the well-known theorem of I. I. Privalov is the following

Theorem 1. *If a normal meromorphic function $f(z)$ in the disk D has a definite asymptotic value along some Jordan curve lying in D and having at least two distinct limiting points on the boundary, then $f(z) = \text{const}$.*

Since every regular function in D has at least one definite asymptotic value along some Jordan curve whose limiting points lie on the boundary (2), Theorem 1 immediately implies the following assertion, announced in (3):

For any normal regular function in the disk D one can find on the unit circle a point $e^{i\theta}$ at which this function has a definite angular boundary value.

2. In the above-mentioned result of Lehto and Virtanen and in Theorem 1, the existence of the limit of the function along all points of some curve was assumed. A natural question arises: is it possible, while preserving the assertions, to weaken this assumption by requiring the existence of the limit of the function only along some, in a certain sense "dense," sequence of points $\{z_n\}$ from the disk D, whose limiting points lie on the boundary.

This question was considered in (3) and was answered affirmatively for certain classes of normal functions; moreover, as the "measure of density"of the sequence $\{z_n\}$, depending on the breadth of the classes, the conditions were:

1) $\rho(z_n, z_{n+1}) < M$, where M is some constant independent of n, or

1′) $\lim\limits_{n\to\infty} \rho(z_n, z_{n+1}) = 0$, where

$$\rho = (a, b) = \frac{1}{2} \ln \left|\frac{1 - a\bar{b}}{1 - \bar{a}b}\right| + \left|\frac{a - b}{1 - \bar{a}b}\right|, \quad |a| < 1,\ |b| < 1.$$

Theorem 2. *If: 1) the sequence $\{z_n\}$ of points from the disk $|z| < 1$, $|z_n| \to 1$ as $n \to \infty$, whose limiting points cover some arc A of the unit circle, $\operatorname{mes} A > 0$, is such that for almost all points $\zeta \in A$ from $\{z_n\}$ one can choose at least one subsequence $\{z_n^{(\zeta)}\}$, which converges to ζ, and for all n*

$$\rho(z_n^{(\zeta)}, z_{n+1}^{(\zeta)}) < M(\zeta)$$

with a constant $M(\zeta)$ depending, perhaps, on the choice of the point ζ; 2) for

for a normal meromorphic function $f(z)$ in $|z| < 1$ having at least one exceptional value, there exists $\lim\limits_{n\to\infty} f(z_n) = a$, then almost everywhere on A the function $f(z)$ has angular boundary values equal to a, and, consequently, $f(z) \equiv a$.

Theorem 2 is a simple consequence of a theorem of Noshiro (4) and auxiliary Lemma 1, whose proof, in essence, is contained in (3) and is a consequence of known properties of normal families of functions (5).

Lemma 1. If: 1) the sequences of points $\{z_n\}$, $\{z_n'\}$ from D, $|z_n'| \to 1$, $n \to \infty$, are such that $\rho(z_n, z_n') < M$ with some constant M independent of n; 2) for a normal meromorphic function $f(z)$ in D, omitting the value c, there exists $\lim\limits_{n\to\infty} f(z_n) = c$, then $\lim\limits_{n\to\infty} f(z_n') = c$.

3. Let us note some further consequences of Theorem 1, Lemma 1, and the assertion corresponding to it in (3).

Theorem 3. If: 1) a sequence of points $\{z_n\}$ from D, having two distinct limit points on the boundary, is such that $\rho(z_n, z_{n+1}) < M$ with some M independent of n; 2) a normal meromorphic function $f(z)$, omitting the value c, has $\lim\limits_{n\to\infty} f(z_n) = c$, then $f(z) \equiv c$.

Theorem 4. If: 1) a sequence of points $\{z_n\}$ from D, having two distinct limit points on the boundary, is such that $\lim\limits_{n\to\infty} \rho(z_n, z_{n+1}) = 0$; 2) a normal meromorphic function $f(z)$ has $\lim\limits_{n\to\infty} f(z_n) = a$, finite or infinite, then $f(z) \equiv a$.

4. If one introduces the concept of a normal quasiconformal mapping (6), then, taking into account the estimates (7) of the distortion of the hyperbolic metric under one-to-one quasiconformal mappings, the results stated above are automatically carried over to the case of normal quasiconformal mappings of the disk D onto the corresponding Riemann surfaces.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.