ON BASES IN THE SPACE OF FUNCTIONS ANALYTIC IN THE CLOSED DISK \(|z|\le R\)

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Abstract

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ON BASES IN THE SPACE OF FUNCTIONS ANALYTIC IN THE CLOSED DISK $|z| \leq R$

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In recent papers (^{1,2}) M. M. Dragilev proved a number of results for bases in A_R—the space of functions analytic in the disk $|z| < R$, considered with the topology of uniform convergence on every closed subset of the disk $|z| < R$. In (^{1}) it is shown that every basis in A_R is a basis of regular convergence, i.e., if $x(z) = \sum a_k x_k(z)$ is the basis expansion of an element, then

$$\sum |a_k| \max_{|z| \leq r} |x_k(z)| < \infty$$

for $0 \leq r < R$. In (^{2}) it is shown that every basis in A_R, after suitable normalization and permutation of the elements, becomes the L-image of the power basis, i.e., is obtained from the power basis by applying a one-to-one and bicontinuous linear mapping of A_R onto itself.

In the present article analogous results are proved for $\overline{A}_R$—the space of functions analytic in the closed disk $|z| \leq R$, with the topology of the inductive limit

$$\overline{A}_R = \lim_{s \to \infty} \operatorname{ind} A_{r_s} \qquad (r_s > r_{s+1} > R,\ r_s \to R).$$

In this topology a countable sequence $x_n(z)$ converges to $x(z)$ in $\overline{A}_R$ if and only if there exists an s such that $x_n, x \in A_{r_s}$ and $x_n(z) \to x(z)$ in A_{r_s}.

Up to isomorphism the relations $\overline{A}_R^* = A_{1/R}$, $A_{1/R}^* = \overline{A}_R$ hold exactly, i.e. $\overline{A}_R$ is a reflexive space.

In what follows we shall need the following two propositions, proved by Newns (^{3}).

1°. Let E be a complete countably normed space with topology T. If every point $x \in E$ is representable in the form of a series $\sum f_k(x) x_k$, convergent in a topology T' weaker than the original topology T, then $\{f_k\}$ is a system of linear continuous functionals in E, biorthogonal to $\{x_k\}$.

2°. Let $E = \lim_{s \to \infty} \operatorname{ind} E_s$, where E_s is a complete countably normed space and $E_s \subset E_{s+1}$. Suppose that for some s every point $x \in E_s$ can be represented by a series $\sum a_k x_k$, convergent in the topology of the space E. Then there exists $\sigma(s) > s$ such that the series converges for every $x \in E_s$ in the topology T_σ of the space E_σ.

Lemma 1. Let x_k be a basis in $E = \lim_{s\to\infty} \operatorname{ind} E_s$ and

$$x = \sum f_k(x) x_k \tag{1}$$

representation of any element x in this basis. Then $\{f_k\}$ is a system biorthogonal to $\{x_k\}$.

It remains to show that the f_k are continuous functionals. By 2°, for every $x \in E_s$ the series (1) converges in the topology T_σ of the space E_σ. The imbedding operation of E_s into E_σ is continuous; therefore the topology T_σ is weaker than the topology T_s. Applying 1°, we obtain that the f_k are continuous functionals in E_s. Since s is arbitrary, this means that the f_k are continuous functionals in E.

Lemma 2. *Every basis in $\overline{A}_R$ is a system biorthogonal to some basis in $A_{1/R}$, and conversely, every system biorthogonal to some basis in $A_{1/R}$ is a basis in $\overline{A}_R$.*

Let $\{x_k\}$ be a basis in $\overline{A}_R$; then, by Lemma 1,

$$x = \sum f_k(x) x_k,$$

where $f_k \in \overline{A}_R^* = A_{1/R}$. For every $f \in A_{1/R}$ there is an expansion

$$f(x) = \sum f(x_k) f_k(x),$$

where the series converges for each $x \in \overline{A}_R$. By the reflexivity of $\overline{A}_R$, the two kinds of weak convergence of functionals in $\overline{A}_R$ coincide. Moreover, weak and strong convergence in $A_{1/R}$ coincide. Therefore $\{x_k\}$ is a basis in $A_{1/R}$. The converse assertion is proved analogously.

We shall say that a basis in a linear topological space E is an **unconditional basis** if, after any permutation of its elements, it remains a basis in E.

Theorem 1. *Every basis in $\overline{A}_R$ is an unconditional basis.*

Let $\{x_k\}$ be a basis in $\overline{A}_R$; then, by Lemma 2, the biorthogonal system $\{f_k\}$ is a basis in $A_{1/R}$. But $\{f_k\}$ is a basis of regular convergence in $A_{1/R}$, and, moreover, an unconditional basis. Therefore $\{f_{k_j}\}$ is a basis in $A_{1/R}$ for every permutation k_j of the natural numbers. But then, by Lemma 2, $\{x_{k_j}\}$ is a basis in $\overline{A}_R$, as the system biorthogonal to a basis. This completes the proof.

Theorem 2. *Every basis $\{x_k\}$ in $\overline{A}_R$ is the L-image of the basis obtained from the power basis by a certain permutation of the elements and by multiplying them by certain numbers.*

Take in $A_{1/R}$ a basis $\{f_k\}$ biorthogonal to $\{x_k\}$. It is obtained from the power basis by means of the transformations indicated in the theorem. Let these transformations be, successively,

$$f_k' = z^{n_k}, \qquad f_k'' = t_k f_k', \qquad f_k = M f_k'',$$

where M is a one-to-one and bicontinuous linear mapping of $A_{1/R}$ onto itself. The power basis $\{z^n\}$ in $A_{1/R}$ is biorthogonal to the same power basis $\{z^n\}$ in $\overline{A}_R$. Now carry out over $\{z^n\}$ in $\overline{A}_R$ successively the following transformations:

$$x_k' = z^{n_k}, \qquad x_k'' = \frac{1}{t_k} x_k', \qquad x_k^{(0)} = (M^{-1})^* x_k''.$$

Obviously, $\{x_k'\}$ and $\{x_k''\}$ are biorthogonal respectively to $\{f_k'\}$ and $\{f_k''\}$. Then $\{x_k^{(0)}\}$ is biorthogonal to f_k. Indeed,

$$(f_k, x_j^{(0)}) = (f_k, (M^{-1})^* x_j'') = (M^{-1} f_k, x_j'') = (f_k, x_j'') = \delta_{k,j}.$$

By the uniqueness of the biorthogonal system, the basis $\{x_k\}$ coincides with the basis $\{x_k^{(0)}\}$. The theorem is proved.

We shall call a basis $\{x_k\}$ in $\overline{A}_R$ a basis of **regular convergence** if for every $x \in \overline{A}_R$ there exists $r(x) > 1$ such that

$$\sum |f_k(x)| \max_{|z| \le r(x)} |x_k(z)| < \infty.$$

Theorem 3. Every basis in $\bar{A}_R$ is a basis of regular convergence. It suffices to prove the theorem for bases that are L-images of the power basis. Let $x_k = M(z^k)$ be such a basis. Then, by (5), in view of the continuity of M, for every $r > R$ there exist $\rho(r) > R$ and $C_r > 0$ such that

$$\max_{|z| \le \rho(r)} |x_k(z)| \le C_r \max_{|z| \le r} |z^k| = C_r r^k \qquad (k = 0, 1, \ldots).$$

Now let $x(z)$ be an arbitrary element of $\bar{A}_R$, and let $\sum a_k z^k$ be the power-series expansion of the element $y = M^{-1} x$; moreover, obviously, $\sum |a_k| r^k < \infty$ for $r < r_0$, where r_0 is the radius of convergence of the power series for $y(z)$. On the other hand, $x = \sum a_k x_k$. By (2) we obtain

$$\sum |a_k| \max_{|z| \le \rho(r)} |x_k(z)| \le C_r \sum |a_k| r^k < \infty.$$

This last inequality completes the proof of the theorem.

In conclusion, I consider it my pleasant duty to express my gratitude to my scientific adviser M. G. Khaplanov.

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Note: Figure translations are in progress. See original paper for figures.