DIMITAR SKORDEV

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

DIMITAR SKORDEV

ON SOME PARTIALLY ORDERED SPACES

(Presented by Academician A. N. Kolmogorov on 18 II 1961)

Let, in a locally convex real linear space X, there be given a convex cone K and a continuous linear functional $f_0(x)$, for which the four conditions listed below are satisfied.

Condition I. Every element of the space X can be represented in the form of the difference of two elements of the cone K.

Condition II. $f_0(x) > 0$ for $x \in K,\ x \neq 0$.

Denote by S the set of those $x \in K$ for which $f_0(x) = 1$.

Condition III. The set S is bicompact.

Introduce an order in X by means of the convention: $x \geq y$, if $x - y \in K$. Define a certain set R_c of operators as follows: $a \in R_c$ if $a(x)$ is a linear operator in the space X, which is defined for every $x \in X$, is continuous if considered only for $x \in K$, and for a suitable value of the number α satisfies the inequalities $\alpha x \geq a(x) \geq -\alpha x$ for every $x \in K$. It is easy to verify that R_c is a ring. In R_c we establish an order: if $a \in R_c$ and $b \in R_c$, then $a \geq b$ will mean that $a(x) \geq b(x)$ for every $x \in K$. We shall call a vector x **quasi-indecomposable** if $x \in K,\ x \neq 0$, and for every $a \in R_c$ the vector $a(x)$ is collinear with the vector x. It is not difficult to see that every indecomposable vector is quasi-indecomposable (we call a vector x indecomposable if $x \in K,\ x \neq 0$, and from $x \geq y \geq 0$ it follows that y is collinear with x).

Condition IV. If three quasi-indecomposable vectors are pairwise non-collinear, then they are linearly independent.

As an example, consider the space X of all functions $x(t)$ of bounded variation on the segment $[\lambda, \mu]$, for which $x(\lambda) = 0$ and $x(t) = \frac{1}{2}[x(t-0) + x(t+0)]$ for $\lambda < t < \mu$. The topology in X is determined by means of the system of seminorms

$$p_n(x) = \int_\lambda^\mu t^n \, dx(t), \quad n = 0, 1, 2, \ldots .$$

As K we choose the set of nondecreasing functions from X and put $f_0(x) = x(\mu)$. In this case the operators $y = a(x)$, where

$$y(t)=\int_{\lambda}^{t} u(s)\,dx(s)$$

and $u(s)$ is continuous on $[\lambda,\mu]$, belong to R_c. Using this, one can establish (see (7)) that in the given case a function from K is a quasi-indecomposable vector if and only if it has only one point of increase. Concerning other examples in which operators from R_c are systematically used, see (4–6).

Returning to the general case, denote by M the (obviously nonempty and bicompact) set of those quasi-indecomposable vectors x for which $f_0(x)=1$. Further denote by C the partially ordered ring of all real continuous functions defined on M, and by C^* the partially ordered linear space of all linear functio-

subspaces in C that are bounded with respect to the uniform norm in C (the ordering in C and in C^* is introduced in the usual way).

Theorem 1. For every $\varphi\in C$ there exists an operator $a_\varphi\in R_c$, uniquely determined by the requirement that the equality $a_\varphi(x)=\varphi(x)x$ hold for every $x\in M$. The correspondence $\varphi\to a_\varphi$ is an isomorphism of the semi-ordered ring C onto the semi-ordered ring R_c.

Theorem 2. For every $\Phi\in C^*$ there exists an element $x_\Phi\in X$, uniquely determined by the requirement that the equality $\Phi(\varphi)=f_0(a_\varphi(x_\Phi))$ hold for any $\varphi\in C$. The correspondence $\Phi\to x_\Phi$ is an isomorphism of the semi-ordered linear space C^* onto the semi-ordered linear space X.

We shall indicate the proof of Theorem 1. From the definition of the set M it is clear that for every $a\in R_c$ there exists a function $\varphi(x)$, uniquely defined for any $x\in M$ by the equality $a(x)=\varphi(x)x$. It is not difficult to see that this function is continuous. The correspondence $a\to\varphi$ is a monotone homomorphism of the semi-ordered ring R_c into the semi-ordered ring C. This correspondence is one-to-one. Indeed, if some $a\in R_c$ corresponds to the constant 0, then $a(x)=0$ for every extreme point x of the set S (all extreme points of S belong to M). On the basis of the Krein-Milman theorem (1) we conclude that $a(x)=0$ for any $x\in S$, whence it follows that $a(x)$ is identically equal to 0. Arguing analogously, we see that if some $a\in R_c$ corresponds to a non-negative function φ, then for any $x\in S$ we have $a(x)\in K$, whence it follows that $a\geq 0$. In other words, the correspondence under consideration is an isomorphism of the semi-ordered ring R_c onto some subring C_0 of the semi-ordered ring C. Using condition III, we prove that R_c is complete with respect to the norm $\|a\|$, defined as the smallest of the numbers α for which $\alpha x\geq a(x)\geq-\alpha x$ for any $x\in K$. Hence it follows that C_0 is complete with respect to the uniform norm. Let x_1 and x_2 be two distinct points of M. In view of condition IV the vector x_1+x_2 is not quasi-indecomposable. If we choose $a\in R_c$ in such a way that the vector $a(x_1+x_2)$ is not collinear with x_1+x_2, then for the function φ corresponding to the operator a we shall have $\varphi(x_1)\neq\varphi(x_2)$, i.e. the ring C_0 separates the points of M. Applying a theorem of Stone (^{3}), we obtain that $C_0 = C$, and Theorem 1 is proved.

We pass to the proof of Theorem 2. Let x be a given element of the space X. Consider the linear functional Φ in C, defined by the equality $\Phi(\varphi) = f_0(a_\varphi(x))$. If $x \in K$, then $\Phi(\varphi) \geq 0$ for every non-negative $\varphi \in C$. It follows from this that the correspondence $x \to \Phi$ is a monotone linear mapping of the semi-ordered linear space X into the semi-ordered linear space C^*. Denote this mapping by τ. Suppose that for some $x_0 \in X$ the corresponding functional $\Phi_0(\varphi)$ is identically equal to 0. Assume that $x_0 \neq 0$. Choose a continuous linear functional $f(x)$ in X for which $f(x_0) \neq 0$. Denote by φ the element of C obtained if $f(x)$ is considered only for $x \in M$. By means of the Krein-Milman theorem it is easy to prove that for every $x \in S$ and, consequently, for every $x \in X$, the equality $f_0(a_\varphi(x)) = f(x)$ holds. For $x = x_0$ we obtain $\Phi_0(\varphi) = f(x_0) \neq 0$, which contradicts the assumption. Consequently, $x_0 = 0$. Thus it is proved that τ is a one-to-one correspondence. Denote by Σ the set of all positive linear functionals in C whose value on the constant 1 is equal to 1. It is known that Σ is the smallest weakly closed convex set in C^* that contains all functionals of the form $\Phi(\varphi) = \varphi(x_0)$, where $x_0 \in M$. Let Σ_1 be the image of S under the correspondence τ. The set Σ_1 is weakly closed and convex, since τ is weakly continuous and linear. For every $x_0 \in M$ the functional $\Phi(\varphi) = \varphi(x_0)$ is the image of x_0 under the correspondence τ and, consequently, belongs to Σ_1. This gives $\Sigma_1 \supset \Sigma$, whence it follows that the image of K under the correspondence τ coincides with the set of all positive linear functionals

functionals in C. Since, on the basis of a theorem of Riesz (^{2}), every element of C^* is representable as the difference of two positive linear functionals in C, it follows at once that τ is an isomorphism of the partially ordered linear space X onto the partially ordered linear space C^*. Considering the inverse correspondence, we are convinced of the validity of Theorem 2.

Using the results of Riesz (^{2}), from Theorem 2 we obtain.

Corollary. Every nonempty subset, bounded above, of the partially ordered space X has an exact least upper bound.

In conclusion we note the following: under our assumptions all quasi-indecomposable vectors are indecomposable (which can be seen by using Theorem 2), but this cannot be asserted if condition IV is omitted.

Sofia State University
Sofia, Bulgaria

Received
10 IX 1960

References

[1] M. Krein, D. Milman, *Studia Math.*, **9**, 133 (1940).
[2] F. Riesz, *Atti Congresso Bologna*, **3**, 143 (1928).
[3] M. Stone, *Trans. Am. Math. Soc.*, **41**, 375 (1937).
[4] Ya. Tagamlitski, *Yearbook of Sofia Univ., Phys.-Math. Fac.*, **46**, book 1, 385 (1949-1950).
[5] Ya. Tagamlitski, *Yearbook of Sofia Univ., Phys.-Math. Fac.*, **47**, book 1, part 2, 85 (1950-1951, 1951-1952).
[6] Ya. Tagamlitski, *Yearbook of Sofia Univ., Phys.-Math. Fac.*, **50**, book 1, part 1, 135 (1955-1956).
[7] Ya. Tagamlitski, *Differential and Integral Calculus*, Sofia, 1957, p. 661.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.