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Abstract

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MATHEMATICAL PHYSICS

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ON A POINT INTERACTION FOR A SYSTEM OF THREE PARTICLES IN QUANTUM MECHANICS

(Presented by Academician I. G. Petrovskii, 8 VI 1961)

In a recent note by F. A. Berezin and one of the authors ⁽¹⁾, a mathematically correct description of point interaction in quantum mechanics was proposed. The method proposed was illustrated by the example of the interaction of two nonrelativistic scalar particles and led to formulas repeatedly obtained in the physical literature by means of procedures to which, however, no unambiguous mathematical meaning can be assigned (see, for example, ^(2, 3)). In the present paper this approach is applied to the study of a system of three particles interacting pairwise in a point manner.

1. The energy operator for a system of three particles is defined in the Hilbert space $L^2(E_9)$ of functions $\psi(x_1, x_2, x_3)$ (x_1, x_2, x_3 are vectors ranging over all three-dimensional space) by means of the differential expression:

$$H = -\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2} - \frac{1}{2m_3}\Delta_{x_3} + V_{12}(r_{12}) + V_{13}(r_{13}) + V_{23}(r_{23}), \quad (1)$$

where Δ_{x_i} is the Laplace operator with respect to the variable x_i ($i = 1, 2, 3$); $r_{ij} = x_i - x_j$; $V_{ik}(r)$ are functions describing the pair interactions. In the case of a point interaction the potentials $V_{ik}(r)$ should be regarded as equal to zero for all $r \neq 0$, which leads to the use of singular functions of the δ -function type; moreover, the differential expression ceases to define a self-adjoint operator in $L^2(E_9)$, no matter how we choose its domain of definition. Therefore, instead of introducing singular potentials into the energy operator, the following approach to the description of point interaction is proposed.

Consider the operator \tilde{H}_0 , defined by the differential expression

$$\tilde{H}_0 = -\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2} - \frac{1}{2m_3}\Delta_{x_3} \quad (2)$$

on the set of sufficiently smooth rapidly decreasing functions $\psi(x_1, x_2, x_3)$ satisfying the condition

$$\psi(x_1, x_1, x_3) = \psi(x_1, x_2, x_2) = \psi(x_1, x_2, x_1) = 0. \quad (3)$$

The operator \widetilde{H}_0 is symmetric and, as is not difficult to verify, has nonzero deficiency indices.

The point nature of the interaction means that the true energy operator H on functions satisfying condition (3) coincides with the operator \widetilde{H}_0 and, thus, is contained among its self-adjoint extensions.

Below we describe all self-adjoint extensions of the operator \widetilde{H}_0 and study in more detail one of them, which occurs in the physical literature.

2. For simplicity we shall assume the particles to be identical, i.e., set $m_1 = m_2 = m_3 = \frac{1}{2}$, and we shall study the operator \widetilde{H}_0 and all its extensions in the subspace of completely symmetric functions

$$\psi(x_1, x_2, x_3) = \psi(x_{i_1}, x_{i_2}, x_{i_3}), \quad (4)$$

where (i_1, i_2, i_3) is an arbitrary permutation of the indices $(1, 2, 3)$.

It is convenient to pass to the momentum representation and separate off the motion of the center of inertia of the system, after which our problem reduces to the following:

In the Hilbert space $L^2(E_6)$ of functions $f(k_1, k_2)$ satisfying the conditions

$$f(k_1, k_2) = f(k_2, k_1) = f(k_1, -k_1 - k_2), \quad (5)$$

there is given a closed symmetric nonnegative operator

$$\widetilde{H}_0 f(k_1, k_2) = (k_1^2 + k_1 k_2 + k_2^2) f(k_1, k_2), \quad (6)$$

whose domain $D(\widetilde{H}_0)$ consists of functions $f(k_1, k_2)$ satisfying the conditions

$$\int (k_1^4 + k_2^4) |f(k_1, k_2)|^2 dk_1 dk_2 < \infty, \quad (7)$$

$$\int f(k_1, k_2) dk_2 = 0. \quad (8)$$

It is required to describe all its self-adjoint extensions.

We note that condition (5) is a consequence of the symmetry (4), while condition (8) is equivalent to requirement (3).

3. Since the operator \widetilde{H}_0 is nonnegative, it is natural to use the theory of extensions of semibounded operators.* With the aid of condition (8) it is not difficult to verify that the defect subspace R_{-1} of the operator $\widetilde{H}_0 + E$ consists of elements of the form

$$U_\varphi(k_1, k_2) = \frac{\varphi(k_1) + \varphi(k_2) + \varphi(-k_1, -k_2)}{k_1^2 + k_1 k_2 + k_2^2 + 1}. \quad (9)$$

It follows that the subspace R_{-1} is isomorphic to the Hilbert space \mathfrak{H} of functions of one vector $\varphi(k)$, obtained by completing $L^2(E_3)$ with respect to the scalar product $[\varphi, \psi] = (W\varphi, \psi)$, where (\cdot, \cdot) is the usual scalar product in $L^2(E_3)$ and W is the positive operator with kernel

$$W(k, k') = \frac{3\pi^2}{\sqrt{\frac{3}{4}k^2 + 1}} \delta(k - k') + \frac{6}{(k^2 + kk' + k'^2 + 1)^2}. \quad (10)$$

The domain $D(\widetilde{H}_0^*)$ of the operator \widetilde{H}_0^* consists of elements of the form

$$g(k_1, k_2) = f(k_1, k_2) + U_\varphi(k_1, k_2) + \frac{U_\psi(k_1, k_2)}{k_1^2 + k_1 k_2 + k_2^2 + 1}, \quad (11)$$

where $f(k_1, k_2) \in D(\widetilde{H}_0)$, $\varphi, \psi \in \mathfrak{H}$. The operator \widetilde{H}_0^* acts according to the formula

$$\widetilde{H}_0^* g(k_1, k_2) = (k_1^2 + k_1 k_2 + k_2^2)g(k_1, k_2) - \varphi(k_1) - \varphi(k_2) - \varphi(-k_1 - k_2). \quad (12)$$

Any self-adjoint extension H_A of the operator \widetilde{H}_0 is obtained by narrowing the domain $D(\widetilde{H}_0^*)$ by means of a relation of the type

$$\psi(k) = A\varphi(k), \quad (13)$$

where A is an arbitrary self-adjoint operator in \mathfrak{H} .

* The theory of extensions of semibounded operators was developed by Friedrichs, M. G. Krein, and others. The methods used by us may be found in the work of M. Sh. Birman ⁽⁴⁾.

4. Let us apply the relations written above to construct the resolvent of the operator H_A . From the equation

$$(H_A - zE)g = h, \quad \text{Im } z \neq 0, \quad (14)$$

and formula (12), we find that g has the form

$$g(k_1, k_2) = \frac{h(k_1, k_2) + \varphi(k_1) + \varphi(k_2) + \varphi(-k_1 - k_2)}{k_1^2 + k_1 k_2 + k_2^2 - z}. \quad (15)$$

Comparing (15) with (11) and using conditions (13) and (8), we arrive at an equation for determining $\varphi(k)$ in terms of $h(k_1, k_2)$:

$$\{[T(z) - T(-1)] + WA\}\varphi(k) = \int \frac{h(k, k') dk'}{k^2 + kk' + k'^2 - z} \equiv \chi_h(k). \quad (16)$$

Here, for brevity, we have introduced the notation

$$T(z)\varphi(k) = 2\pi^2 i \sqrt{z - \frac{3}{4}k^2} \varphi(k) + 2 \int \frac{\varphi(k') dk'}{k^2 + kk' + k'^2 - z}, \quad \text{Im } z \neq 0. \quad (17)$$

It is not hard to verify that the expression $T(z) - T(-1)$ defines a bounded operator acting from \mathfrak{H} into \mathfrak{H}' , where \mathfrak{H}' is the Hilbert space with scalar product specified by the operator W^{-1} ; moreover $\chi_h(k) \in \mathfrak{H}'$.

- Several physical works have been devoted to the problem of point interaction of three particles. In the work of K. A. Ter-Martirosyan and G. V. Skornyyakov⁵, the problem of finding the resolvent of the corresponding energy operator is reduced to an equation essentially equivalent to the following:

$$\left(\alpha + 2\pi^2 i \sqrt{z - \frac{3}{4}k^2} \right) \varphi(k) + 2 \int \frac{\varphi(k') dk'}{k^2 + kk' + k'^2 - z} = \chi(k), \quad (18)$$

where α is a certain real number characteristic of the model. Such an equation can be obtained formally from our equation (16) if we put

$$A = W^{-1}[T(-1) + \alpha E]. \quad (19)$$

The requirement that such an A be self-adjoint in \mathfrak{H} is equivalent to the self-adjointness of $T(-1)$ in $L^2(E_3)$. Note, however, that up to now we have avoided assigning a direct meaning to the operator $T(z)$ and, in particular, $T(-1)$. This caution is not accidental, since the operator $T(z)$ is unbounded in $L^2(E_3)$. Denote by $\tilde{T}(z)$ the operator defined by expression (17) on finite functions. It can be shown that the operator $\tilde{T}(-1)$ is a symmetric operator and has deficiency

indices $(1, 1)$.^{*} The domain of definition of its adjoint $T^*(-1)$ contains, in addition to the elements from the domain of definition of the closure of $\tilde{T}(-1)$, also functions of the form

$$\varphi(k) = \frac{1}{k^2 + 1} (c_1 \sin \lambda_0 \ln |k| + c_2 \cos \lambda_0 \ln |k|), \quad (20)$$

where λ_0 is the unique positive root of the transcendental equation

$$1 - \frac{8}{\sqrt{3}} \frac{\operatorname{sh} \frac{\pi\lambda}{6}}{\lambda \operatorname{ch} \frac{\pi\lambda}{2}} = 0 \quad (21)$$

and c_1 and c_2 are arbitrary complex numbers. For any self-adjoint—

^{*} The latter circumstance is connected with the fact that the homogeneous equation (18), as G. S. Danilov⁶ observed, has nontrivial solutions in $L^2(E_3)$ for all z .

of the extension $\tilde{T}(-1)$ one must set

$$c_1 = \beta c_2 \quad (\beta \text{ real}). \quad (22)$$

We shall denote the corresponding extension by $\tilde{T}_\beta(-1)$. If in formula (19) under $T(-1)$ we understand the fixed extension $\tilde{T}_\beta(-1)$, then we arrive at the equation

$$(\tilde{T}_\beta(z) + \alpha E)\varphi = \chi, \quad (23)$$

where the operator $\tilde{T}_\beta(z)$ has the same domain of definition as $\tilde{T}_\beta(-1)$. In other words, among the solutions of equation (18) we select those solutions which, for large k , have the asymptotics

$$\varphi(k) = \frac{c}{k^2} [\beta \sin \lambda_0 \ln k + \cos \lambda_0 \ln k] + o\left(\frac{1}{k^2}\right). \quad (24)$$

In the work of G. S. Danilov the same prescription is proposed for selecting physically interesting solutions of the Ter-Martirosyan-Skorniyakov equation. Thus, our general scheme contains the model of point interaction of Ter-Martirosyan and Skorniyakov, as refined by Danilov.

6. In conclusion we make several remarks. It can be shown that the continuous spectrum of the extension $H_{\alpha,\beta}$ described in the preceding section lies to the right of the point $-\alpha^2$ for $\alpha > 0$ and to the right of 0 for $\alpha < 0$. The detailed structure of the eigenfunctions of the continuous spectrum will be studied later. In addition to the continuous spectrum, the operator $H_{\alpha,\beta}$ has a countable set of discrete eigenvalues E_n , tending to $-\infty$ with the following asymptotics as $n \rightarrow \infty$:

$$E_n = -3 \exp \left[\frac{2\pi n}{\lambda_0} - \frac{2}{\lambda_0} \operatorname{arc\,tg} \frac{1}{\beta} \right] [1 + o(1)]. \quad (25)$$

We note that the last result somewhat discredits the chosen extension, since, probably, in nonrelativistic quantum mechanics only semibounded energy operators are of interest. It seems to us that among other extensions of the operator \widetilde{H}_0 there exist semibounded extensions possessing all the properties of the Ter-Martirosyan and Skorniyakov model that are good from the physical point of view, namely locality and the correct character of the continuous spectrum. Apparently, such extensions will be obtained if, as the operator A , one chooses

$$A = W^{-1}[T(-1) + \alpha E + K], \quad (26)$$

where K is an integral convolution operator with kernel $K(k - k')$, and $K(\xi)$ has, as $\xi \rightarrow \infty$, the asymptotics

$$K(\xi) \sim \frac{\gamma}{\xi^2}, \quad \gamma > \frac{1}{\pi^3} \left(\frac{4\pi}{3\sqrt{3}} - 1 \right). \quad (27)$$

A detailed development of this point of view is not given for lack of space.

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