MATHEMATICS

N. I. AKHIEZER and Yu. Ya. TOMCHUK

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. I. AKHIEZER and Yu. Ya. TOMCHUK

ON THE THEORY OF ORTHOGONAL POLYNOMIALS ON SEVERAL INTERVALS

(Presented by Academician S. N. Bernstein, 21 I 1961)

1. The present paper is devoted to a further study of polynomials orthogonal on the system of intervals (E)

$$[-1, \alpha_1],\ [\beta_1, \alpha_2], \dots, [\beta_\rho, 1],$$

which was begun in the article (^{1}). In that article the z-plane, cut along E, was denoted by $\mathfrak{G}$, its second copy by $\mathfrak{G}'$, and the Riemann surface constructed from them by $\mathfrak{F}$. If c is a point on $\mathfrak{G}$, then the point $\mathfrak{G}'$ lying "under it" is denoted by c', and conversely. Further,

$$S(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_\rho),$$

$$\sqrt{R(z)} = \sqrt{(z+1)(z-\alpha_1)(z-\beta_1)\cdots(z-1)},$$

where at the point $x > 1$ on $\mathfrak{G}$ the radical has positive value. Finally, polynomials of degree n with leading coefficient equal to 1, orthogonal with respect to the weights

$$\frac{S(x)}{\sqrt{-R(x)}}\frac{1}{t(x)}, \qquad \frac{\sqrt{-R(x)}}{S(x)}\frac{1}{t(x)} \quad (x \in E),$$

are denoted, respectively, by $T_n(x;t)$, $U_n(x;t)$.

In (^{1}) the function

$$p(z, \sqrt{R(z)}) = T_n(z;P) - \frac{\sqrt{R(z)}}{S(z)} U_{n+1}(z;P),$$

was considered, where $P(z)$ is a polynomial of even degree $\rho < n$, positive on E, and it was shown that all poles of this function, as well as all zeros except ρ of them (these ρ zeros were called arbitrary), are known in advance. We shall henceforth assume that the polynomial $P(z)$ is positive not only on E, but also on the whole interval $[-1,1]$. Under this assumption the arbitrary zeros of the function $p(z,\sqrt{R(z)})$ lie in the intervals $[\alpha_k,\beta_k]$, one in each, with some on the sheet $\mathfrak{G}$ (we shall denote them by $\gamma_1,\gamma_2,\dots,\gamma_\lambda$), and the remaining ones on the sheet $\mathfrak{G}'$ (we denote them by $\gamma'_{\lambda+1},\gamma'_{\lambda+2},\dots,\gamma'_\rho$). Further, let, as in (^{1}), $a_1,a_2,\dots,a_\rho$ denote the points on the sheet $\mathfrak{G}$ at which the polynomial $P(z)$ vanishes.

2. From the parameters a_j,γ_k,γ'_i one can construct the function $p(z,\sqrt{R(z)})$, and hence also the orthogonal polynomials $T_n(x,P)$, $U_{n-1}(x,P)$. This is done by means of Abelian integrals belonging to $\mathfrak{F}$. Namely, put

$$h(z)=\exp\left\{\int_1^z \frac{M(z)}{\sqrt{R(z)}}\,dz\right\},$$

$$h(z;c)=\exp\left\{\int_1^z\left[\frac{\sqrt{R(z)}+\sqrt{R(c)}}{z-c}+M_c(z)\right]\frac{dz}{2\sqrt{R(z)}}\right\},$$

where c is an arbitrary finite point of the surface $\mathfrak{F}$, and $M(z), M_c(z)$ are polynomials of degree ρ in z with leading coefficients equal to 1, which are determined by the requirement that the functions $h(z)$, $h(z;c)$ have a single-valued modulus on $\mathfrak{F}$.[*] The only (and simple) pole for each of the functions $h(z)$, $h(z;c)$ the point $z=\infty$ (on $\mathfrak{G}$) is a pole; the only zero (also simple) for the function $h(z)$ is the point $z=\infty'$, and for the function $h(z;c)$, the point c. We note that the quantity

$$\lim_{z\to\infty}\frac{z}{h(z)}=\lim_{z\to\infty'} zh(z)=\exp\left\{\int_1^\infty\left[\frac{1}{x}-\frac{M(x)}{\sqrt{R(x)}}\right]dx\right\}=\tau$$

is the transfinite diameter of the set E. The polynomial $M(z)$ enters into yet another important functional, which may be called the mean geometric value of a (positive) function prescribed on E. Let the function $\varphi(x)$ $(x\in E)$ be positive and continuous. In that case there exists one and only one function $\Phi(z)$, regular in the domain $\mathfrak{G}$ and nonzero there, whose modulus is single-valued, continuous up to the boundary of $\mathfrak{G}$, and satisfies on it the relation $|\Phi(x)|=\varphi(x)$ $(x\in E)$.

For example, to determine the coefficients of the polynomial $M(z)$ one obtains the system of equations

$$\int_{\alpha_k}^{\beta_k}\frac{M(z)}{\sqrt{R(z)}}\,dz=0\qquad(k=1,2,\dots,\rho).$$

The value of this modulus at the point $z = \infty$ is precisely the functional just mentioned. It is expressed by means of the polynomial $M(z)$ in the form

$$|\Phi(\infty)| = \exp\left\{\frac{1}{\pi}\int_E \frac{M(x)}{\sqrt{-R(x)}} \ln\varphi(x)\,dx\right\} \equiv \mathfrak{G}[\varphi(x)],$$

where $\sqrt{-R(x)}$ has positive value on the interval $(\beta_\rho, 1)$.*

The parametric representation mentioned above for the function $p(z, \sqrt{R(z)})$ has the form

$$p(z, \sqrt{R(z)}) = \frac{A}{[h(z)]^n}\prod_{j=1}^{\rho} h(z;\alpha_j)\left[\prod_{k=1}^{\rho} h(z;\alpha_k)\right]^{-1}\prod_{k=1}^{\rho} h(z;\gamma_k)\prod_{j=\lambda+1}^{\rho} h(z;\gamma_k^i), \tag{1}$$

where A is a positive constant and, as indicated above, $n > \rho$. To compute the constant A one uses the equality

$$\lim_{z\to\infty'} z^{-n} p(z, \sqrt{R(z)}) = 2.$$

Then one can compute the quantity

$$N_n[P] = \lim_{z\to\infty}\frac{p(z, \sqrt{R(z)})z^n}{P(z)},$$

which is the normalization coefficient, namely:

$$N_n[P] = \frac{1}{\pi}\int_E [T_n(x;P)]^2 \frac{S(x)}{\sqrt{-R(x)}}\frac{dx}{P(x)} = \frac{1}{\pi}\int_E [U_{n-1}(x;P)]^2 \frac{\sqrt{-R(x)}}{S(x)}\frac{dx}{P(x)}.$$

---

* For $\rho = 0$, i.e., in the case when E is the interval $[-1, 1]$, this functional takes the form

$$\exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\ln\varphi(x)}{\sqrt{1-x^2}}\,dx\right\}$$

and was first introduced by G. Szegő (2). We also note that in the case when

$$\int_E \ln\varphi(x)\frac{x^k\,dx}{\sqrt{-R(x)}} = 0 \qquad (k = 0, 1, \dots, \rho-1), \tag{*}$$

the function $\Phi(z)$ is constructed very simply, namely:

$$\Phi(z) = \exp\left\{ \frac{1}{\pi} \int_E \frac{\sqrt{R(z)}}{\sqrt{-R(\xi)}} \frac{\ln \varphi(\xi)}{z-\xi} \, d\xi \right\}.$$

The right-hand side of this formula represents the operator $A[z; f(x)]$ introduced in article (^{1}), if one puts $f(x) = [\varphi(x)]^2$. Finally, we note that if condition $(*)$ is satisfied, the functional $\mathfrak{G}[\varphi(x)]$ can be replaced by the polynomial $S(x)$, which enters into the functional $G[f(x)]$ of article (^{1}).

The final expressions for the functional $N_n[P]$ and the constant A have the form

$$N_n[P] = 2\tau^{2n}\, \mathfrak{G}\left[\frac{1}{P(x)}\right] \Gamma(\gamma_1, \gamma_2, \dots, \gamma_\rho), \quad A = 2\tau^n\, \mathfrak{G}\left[\sqrt{\frac{P(1)}{P(x)}}\right] \Gamma^*(\gamma_1, \gamma_2, \dots, \gamma_\rho),$$

where the last factors depend only on the arguments displayed, and one can indicate such a finite constant $L > 1$ that always

$$\frac{1}{L} < \Gamma(\gamma_1, \gamma_2, \dots, \gamma_\rho) < L, \quad \frac{1}{L} < \Gamma^*(\gamma_1, \gamma_2, \dots, \gamma_\rho) < L.$$

It follows from representation (1) that for any $x \in E$ and any $n > \rho$,

$$\left|\sqrt{S(x)}\, T_n(x; P)\right| < C\sqrt{P(x)}\sqrt{N_n[P]},$$

where C depends only on the set E.

3. Let us now take an arbitrary continuous positive function $t(x)$ $(x \in E)$ and consider the difference

$$T_n(x; t) - T_n(x; P) = D_n(x), \tag{2}$$

where $P(x)$ is a polynomial positive on the interval $[-1, 1]$, of even degree $p < n$, which we shall dispose of later. By means of simple transformations we find that

$$D_n(x) = \int_E T_n(\xi, t) \frac{K_n(x, \xi)}{x - \xi} \left[\frac{1}{P(\xi)} - \frac{1}{t(\xi)}\right] \frac{S(\xi)\, d\xi}{\sqrt{-R(\xi)}}, \tag{3}$$

where the kernel $K_n(x, \xi)$ is a polynomial of degree n in both variables, equal to zero for $x = \xi$, and satisfying the inequality

$$\left|\sqrt{S(x)}\sqrt{S(\xi)} K_n(x, \xi)\right| < C\sqrt{P(x)}\sqrt{P(\xi)} \quad (x \in E,\ \xi \in E),$$

where C depends only on the set E.

Theorem 1. *If a positive function $t(x)$ $(x \in E)$ is continuously differentiable and the modulus of continuity $\omega_1(\delta)$ of its first derivative satisfies the condition*

$$\lim_{n\to\infty} \omega_1\left(\frac{1}{n}\right)\ln n = 0,$$

then, for all sufficiently large n and any $x \in E$,

$$\left|\sqrt{S(x)}\,T_n(x,t)\right| < C\tau^n\sqrt{t(x)}\,\mathfrak{S}[1/\sqrt{t(x)}], \tag{4}$$

where C is a constant depending only on the set E.

For the proof of this theorem, let us note that for any natural n one can construct polynomials $P_i(x)$ of degree $p = 2\left[\dfrac{n-1}{2}\right]$, satisfying, for every $x \in E$, the inequalities

$$|t(x) - P_0(x)| \le K\left\{\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right\}\omega_1\left(\frac{1}{n}\right),$$

$$|t(x) - P_i(x)| \le K\left\{\frac{\sqrt{(x-\alpha_i)(x-\beta_i)}}{n} + \frac{1}{n^2}\right\}\omega_1\left(\frac{1}{n}\right) \quad (i = 1, 2, \dots, \rho),$$

where K is an absolute constant. The existence of these polynomials follows from a result of A. F. Timan $(^3)$, refining Jackson’ s theorem. Suppose, for example, that we want to prove inequality (4) in the intervals $\left[\frac{1}{2}(\beta_{k-1}+\alpha_k), \alpha_k\right]$, $\left[\beta_k, \frac{1}{2}(\beta_k+\alpha_{k+1})\right]$. In that case one should take in formula (2), as the polynomial $P(x)$, the polynomial $P_k(x)$. Applying the estimates usual in such cases for integrals of the form (3), we find,

that in each of the intervals mentioned

$$\left|\sqrt{S(x)}D_n(x)\right| < C\omega_1\left(\frac{1}{n}\right)\ln n \cdot \sqrt{t(x)}\sup_{x\in E}\frac{\left|\sqrt{S(x)}T_n(x;t)\right|}{\sqrt{t(x)}}.$$

Hence the assertion of the theorem is obtained without any difficulty.

Lemma. Let, for the continuous positive function $t(x)$ $(x \in E)$, there exist a polynomial $p_n(x)$ of even degree $< n$, positive for $-1 \le x \le 1$, and such that everywhere on E

$$\left|1 - \frac{p_n(x)}{t(x)}\right| < \left[\frac{\sqrt{|R(x)|}}{n} + \frac{1}{n^2}\right]\frac{\varepsilon_n}{\ln n}.$$

In that case, everywhere on E,

$$|T_n(x;t) - T_n(x;p_n)| < C\sqrt{p_n(x)}\sqrt{N_n[p_n]}\,\varepsilon_n,$$

where the constant C depends only on the set E.

With the aid of this lemma and the main result of paper (^{1}), the following is proved:

Theorem 2. Let the positive function $t(x)$ $(x \in E)$ have a continuous second derivative, whose modulus of continuity satisfies the relation $\lim_{n\to\infty} \omega_2\left(\frac{1}{n}\right)\ln n = 0$. Let, further, $P(x)$ be some polynomial of even degree, positive on the interval $[-1, 1]$, and such that

$$\int_E \ln t(x)\frac{x^k\,dx}{\sqrt{-R(x)}} = \int_E \ln P(x)\frac{x^k\,dx}{\sqrt{-R(x)}} \qquad (k = 0, 1, 2, \dots, \rho - 1).$$

In that case, as $n \to \infty$, uniformly on E the following asymptotic equality holds:

$$\frac{T_n(x;t)}{\sqrt{t(x)}\sqrt{N_n^*[t]}} \sim$$

$$\sim \frac{1}{\sqrt{P(x)}\sqrt{N_n[P]}}\left\{T_n(x;P)\cos\psi(x) - \frac{\sqrt{-R(x)}}{S(x)}U_{n-1}(x;P)\sin\psi(x)\right\},$$

where

$$N_n^*[t] = N_n[P]\,\mathfrak{S}[P(x)/t(x)]$$

and, as $n \to \infty$,

$$N_n^*[t] \sim N_n[t] = \frac{1}{\pi}\int_E |T_n(x;t)|^2\frac{S(x)}{\sqrt{-R(x)}\,t(x)}\,dx,$$

while $\psi(x)$ is determined by the formula

$$\psi(x) = \frac{1}{2\pi}\,\text{V.p.}\int_E \frac{\sqrt{-R(x)}}{\sqrt{-R(\xi)}}\frac{\ln\dfrac{t(\xi)}{P(\xi)}}{x - \xi}\,d\xi.$$

Kharkov State University
named after A. M. Gorky

Received
19 I 1951

REFERENCES

[1] N. I. Akhiezer, DAN, 134, No. 1 (1960). [2] G. Szegő, *Orthogonal polynomials*, 1939. [3] A. F. Timan, DAN, 78, No. 1 (1951).

Note: Figure translations are in progress. See original paper for figures.