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Abstract

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ON SOME HOMOGENEOUS BOUNDARY-VALUE PROBLEMS

FOR THE EQUATION OF VIBRATIONS OF A STRING

(Presented by Academician S. L. Sobolev on 15 II 1961)

The investigation of the spectral properties of operators generated by the simplest system of the type of S. L. Sobolev leads to the consideration of the following boundary-value eigenvalue problem:

Determine for which values of λ the equation

$$u_{xx} - \lambda^2 u_{yy} = 0, \tag{1}$$

considered in some domain D with boundary Γ, has nontrivial solutions satisfying the condition

$$u\big|_{\Gamma} = 0, \tag{2}$$

and describe the closure of the linear span of these solutions in one or another metric.

Problem (1)–(2) has been studied in detail in the works of R. A. Aleksandryan ($^{1-3}$).

In the present paper equation (1) is considered under the boundary condition

$$Lu\big|_{\Gamma} = 0, \tag{3}$$

where the operator L has the form

$$L = \sum_{i+j=0}^{n} c_{ij} \frac{\partial^{i+j}}{\partial x^i \partial y^j}, \tag{4}$$

c_{ij} are constants, $c_{00} \neq 0$, and n is some natural number. We note that the boundary-value problem under consideration has two special features: first, it pertains to a hyperbolic equation, and second, derivatives of arbitrarily high orders may occur in the boundary condition.

A number λ for which there exists a function $u_\lambda(x, y)$ satisfying (1)–(3), and $Lu_\lambda(x, y) \neq 0$, will be called an eigenvalue of problem (1)–(3).

It is proved that if D is the unit disk with center at the origin, then the eigenfunctions of problem (1)–(3) form a complete system in $L_2(D)$.

Boundary conditions of the form

$$L_1 u\big|_\Gamma = \left[cu + \frac{\partial u}{\partial n}\right]_\Gamma = 0 \tag{3*}$$

are also considered.

It is proved that the positivity of the constant c is sufficient for the eigenfunctions of problem (1)–(3*) to be complete in the class of continuous functions.

It turns out that problem (1)–(3) is closely connected with problem (1)–(2); namely, the eigenvalues of problem (1)–(2) are simultaneously

and with the eigenvalues of problem (1)–(3), and conversely. Indeed, suppose that for some domain D, λ is an eigenvalue, and $T_\lambda(x, y) = T_1(t) + T_2(z)$, where $t = y + \lambda x$; $z = y - \lambda x$, is an eigenfunction of problem (1)–(2).

Then the eigenfunction of problem (1)–(3) will have the form

$$u_\lambda(x, y) = f_1(t) + f_2(z),$$

where $f_1(t)$ and $f_2(z)$ are any solutions of the corresponding equations

$$\sum_{i+j=0}^{n} c_{ij} \lambda^i \frac{d^{i+j} f_1(t)}{dt^{i+j}} = T_1(t); \tag{5}$$

$$\sum_{i+j=0}^{n} c_{ij} (-\lambda)^i \frac{d^{i+j} f_2(z)}{dz^{i+j}} = T_2(z). \tag{6}$$

Such a construction of eigenfunctions of problem (1)–(3), when $c_{00} = 0$, is obviously impossible only for those λ for which all the coefficients in one of the equations (5) or (6) are equal to zero.

If, however, for equation (1) one considers a boundary condition of the form

$$L^*u\big|_\Gamma = \left[\sum_{i+j=0}^{n} c_{ij}\frac{\partial^{i+j}u}{\partial x^i \partial y^j} + \sum_{i=0}^{n} b_i\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^i u\right]_\Gamma = 0 \quad (n > m),$$

then, under certain conditions on c_{ij} $(i+j > m)$, by the same constructions one can show that the eigenvalues of this problem coincide with the eigenvalues of problem (1)–(2).

Theorem 1. *Let the domain D be the circle $x^2 + y^2 = 1$. Then the eigenfunctions of problem (1)–(3) are complete in $L_2(D)$.*

In paper (^{2}) an explicit form was found for the eigenfunctions, polynomial in the degrees of t and z,

$$T_\lambda(x, y) = T_1(t) + T_2(z)$$

of problem (1)–(2), considered in D, and the completeness of these eigenfunctions in $C_0(D)$ was proved. The proof of the completeness of the eigenfunctions of problem (1)–(3) carried out in the present paper is a development of the analogous arguments of paper (^{2}).

Since $T_1(t)$ and $T_2(z)$ are polynomials in powers of t and z, it is known that equations (5) and (6) admit polynomial solutions $f_1(t)$ and $f_2(z)$. Their sum

$$P_\lambda(x, y) = f_1(t) + f_2(z)$$

is a polynomial solution of problem (1)–(3) in D.

Fix $\varepsilon > 0$. Let $\varphi(x, y) \in C_0(D)$. There exists (^{2}) a polynomial $Q(x, y)$, equal to zero on Γ, such that

$$\max_{x^2+y^2\le 1} |\varphi(x, y) - Q(x, y)| < \frac{\varepsilon}{2}.$$

Let $k > n$ be an odd number. Then take

$$R(x, y) = \{[x^2 + y^2 - 1]^k + 1\}^p Q(x, y),$$

where p is an integer greater than zero. Denote $[x^2 + y^2 - 1]^k + 1$ by $R^*(x, y)$. Since for any p, $\{R^*(x, y)\}^p_\Gamma = 1$ and every derivative of $R^*(x, y)$ up to order n on Γ vanishes, we have

$$LR(x, y)\big|_\Gamma = LQ(x, y)\big|_\Gamma.$$

Since $Q(x, y)|_\Gamma = 0$ and $|R(x, y)| \le |Q(x, y)|$ in D, one can choose $\delta > 0$ such that $|R(x, y)|$ will be less than $\varepsilon/2$ in the ring $1 - \delta \le x^2 + y^2 \le 1$. In turn, in the disk of radius $r = \sqrt{1-\delta}$ we have everywhere $|R^*(x,y)| < 1-\delta^k$, and, consequently, p can be chosen so large that $\{R^*(x,y)\}^p < \frac{\varepsilon}{2M}$, where $M = \max\limits_{x^2+y^2\le 1} |Q(x,y)|$. Thus,

$$\max_{x^2+y^2\le 1} |R(x,y)| < \frac{\varepsilon}{2}. \tag{7}$$

We have

$$\max_{x^2+y^2\le 1} |\varphi(x,y) - Q(x,y) + R(x,y)| < \varepsilon. \tag{8}$$

Next, $L\{Q(x,y) - R(x,y)\} = \Theta(x,y)$, and by the property of $R(x,y)$ the polynomial $\Theta(x,y)$ vanishes on Γ. From paper $(^2)$ it follows that $\Theta(x,y)$ is a linear combination of the eigenfunctions $T_\lambda(x,y)$ of problem (1)–(2):

$$\Theta(x,y) = \sum_{k=1}^{N} \alpha_k T_k(x,y).$$

By virtue of (5)–(6),

$$L\left\{\sum_{k=1}^{n} \alpha_k P_k(x,y)\right\} = \sum_{k=1}^{N} \alpha_k T_k(x,y) = \Theta(x,y),$$

where $P_k(x,y)$ are the eigenfunctions of problem (1)–(3). Thus,

$$L\left\{Q(x,y) - R(x,y) - \sum_{k=1}^{N} \alpha_k P_k(x,y)\right\} \equiv 0.$$

But if $\mathcal{P}(x,y)$ is a polynomial and $L\mathcal{P}(x,y) \equiv 0$, then, since $c_{00} \neq 0$, $\mathcal{P}(x,y) \equiv 0$. Therefore

$$Q(x,y) - R(x,y) = \sum_{k=1}^{N} \alpha_k P_k(x,y).$$

Taking (8) into account, we obtain

$$\max_{x^2+y^2\le 1} \left| \varphi(x,y) - \sum_{k=1}^{N} \alpha_k P_k(x,y) \right| < \varepsilon,$$

and since the set of continuous functions that vanish on the boundary is complete in $L_2(D)$, Theorem 1 is proved.

Theorem 2. Let the domain D be the disk $x^2 + y^2 \leq 1$. Then, if $c \neq 0, -1, \dots, -k, ...$, the eigenvalues of problem (1)–(2) are also eigenvalues of problem (1)–(3*), and the eigenfunctions (1)–(3*) are complete in D in the class of continuous functions in the sense of uniform convergence.

Let λ be an eigenvalue of problem (1)–(2). Consider the equations

$$tf_1'(t) + cf_1(t) = T_1(t); \tag{5*}$$

$$zf_2'(z) + cf_2(z) = T_2(z), \tag{6*}$$

where, as before, $T_1(t) + T_2(z)$ is a polynomial eigenfunction of problem (1)–(2), corresponding to the eigenvalue λ. Since $c \neq 0, -1, \dots, -k, ...$, equations (5*) and (6)* admit polynomial-

solutions $f_1(t)$ and $f_2(z)$, whose sum $P_\lambda(x, y) = f_1(t) + f_2(z)$ is a solution of problem (1)–(3*), since

$$\left[cP_\lambda(x, y) + \frac{\partial P_\lambda(x, y)}{\partial n}\right]_\Gamma =$$

$$= cP_\lambda(x, y)\big|_\Gamma + \left[\frac{\partial P_\lambda(x, y)}{dx}\cos(\widehat{nx}) + \frac{\partial P_\lambda(x, y)}{dy}\cos(\widehat{ny})\right]_\Gamma =$$

$$= cP_\lambda(x, y)\big|_\Gamma + \left[x\frac{\partial P_\lambda(x, y)}{dx} + y\frac{\partial P_\lambda(x, y)}{dy}\right]_\Gamma =$$

$$= c\left[f_1(t) + f_2(z)\right]_\Gamma + \left[tf_1'(t) + zf_2'(z)\right]_\Gamma = \left[T_1(t) + T_2(z)\right]_\Gamma = 0.$$

Let $g(x, y)$ be a continuous function in D. Fix $\varepsilon > 0$. There exists a polynomial

$$Q(x, y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j$$

such that

$$\max_{x^2+y^2 \leq 1} |g(x, y) - Q(x, y)| < \frac{\varepsilon}{2}.$$

It is easy to see that the polynomial

$$R(x, y) = \sum_{i+j=0}^{n} \frac{c + i + j}{c + i + j + 2p} a_{ij}(x^2 + y^2)^p x^i y^j$$

for sufficiently large p has the following properties:

$$L_1 R(x,y)\big|_\Gamma = L_1 Q(x,y)\big|_\Gamma; \qquad \max_{x^2+y^2\le 1} |R(x,y)| < \frac{\varepsilon}{2}.$$

The remainder of the proof is carried out exactly as in Theorem 1. An analogous theorem can also be proved for a boundary condition of the form

$$L_1^* u(x,y)\big|_\Gamma = \sum_{i=0}^{n} c_i \left.\frac{\partial^i u(x,y)}{\partial n^i}\right|_\Gamma = 0 \qquad (n \ge 1),$$

if, for every $k = 0, 1, 2, ...$, the following expression is nonzero:

$$c_0 + kc_1 + k(k-1)c_2 + \cdots + \frac{k!}{(k-n)!} c_n.$$

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Note: Figure translations are in progress. See original paper for figures.