N. V. EFIMOV and E. G. POZNYAK

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. V. EFIMOV and E. G. POZNYAK

A GENERALIZATION OF HILBERT' S THEOREM ON SURFACES OF CONSTANT NEGATIVE CURVATURE

(Presented by Academician P. S. Aleksandrov on 26 X 1960)

No. 1. Let Λ be a two-dimensional simply connected manifold on which coordinates (u, v) and a Gaussian metric ds^2 are given, with coefficients $E, F, G \in C^4(u, v)$. We say that Λ is **regularly immersed** in three-dimensional Euclidean space E_3 if there are three functions $L, M, N \in C^2(u, v)$ which, as the coefficients of the second quadratic form of the surface, satisfy throughout Λ the fundamental equations of the theory of surfaces with the given ds^2. Denote by K the Gaussian curvature of the element ds^2; assuming $K < 0$, put $k^2 = -K$. Suppose that on Λ the inequalities

$$k^2 \geqslant 1, \qquad \left|\frac{k'}{k}\right| \leqslant \frac{1}{\Delta}, \qquad \frac{1}{k^{3/2}}\left(\frac{1}{k^{1/2}}\right)''_g \leqslant 1 - h, \tag{1}$$

hold, where k' is the derivative in an arbitrary direction; the second derivative is taken along the arc of an arbitrary geodesic; h and Δ are positive constants.

Theorem A. *If*

$$\frac{1}{\Delta^2} < \frac{4}{3} h \tag{2}$$

and if Λ with the given ds^2 is a complete metric space, then Λ admits no regular immersion in E_3.

Example.

$$ds^2 = du^2 + \left(\operatorname{ch}\sqrt{\frac{3}{2}}\, u + a \sin u\right)^2 dv^2;$$

for sufficiently small a all the conditions of Theorem A are fulfilled.

Theorem B. *In E_3 there is no complete regular surface whose Gaussian curvature is negative and satisfies conditions (1), (2).*

Theorem B follows from Theorem A. Obviously, Theorem B includes, as a special case, the well-known Hilbert theorem on surfaces of constant negative curvature.

No. 2. To prove Theorem A by contradiction, suppose that Λ is regularly immersed in E_3; then on Λ a locally regular asymptotic net is defined. Let (u, v) be local asymptotic coordinates (we regard the family $v = \text{const}$ as the first); let s_1, s_2 be the natural parameters of the asymptotic curves; and let ω be the angle between the asymptotics, $0 < \omega < \pi$.

Lemma 1. *The set of values of* $\int_L \sin\omega\, ds$, *over the totality of all asymptotic arcs* L, *cannot be bounded.*

The proof is based on the formulas:

$$\frac{\partial \ln(ek)}{\partial s_2} = \sin\omega \frac{\partial Q}{\partial s_1^*}, \qquad \frac{\partial \ln(gk)}{\partial s_1} = -\sin\omega \frac{\partial Q}{\partial s_2^*}, \tag{3}$$

$$\frac{1}{\rho_1} = -\frac{\partial \omega}{\partial s_1} + \sin\omega \frac{\partial Q}{\partial s_2}, \qquad \frac{1}{\rho_2} = \frac{\partial \omega}{\partial s_2} - \sin\omega \frac{\partial Q}{\partial s_1}, \tag{4}$$

where $e^2 = E,\ g^2 = G$ in asymptotic coordinates; $Q = \frac{1}{2}\ln k$; $\partial/\partial s_1^*,\ \partial/\partial s_2$ are symbols of differentiation in directions orthogonal to the asymptotics; $1/\rho_1,\ 1/\rho_2$ are the geodesic curvatures of the asymptotics.

From relations (3), (4) and from the assumption

$$\int_L \sin\omega\, ds \le C = \text{const}$$

it follows that each asymptotic of one family intersects every asymptotic of the other, and then a contradiction is obtained with the Gauss-Bonnet formula.

No. 3. There is a proposition, stated in general outline by N. V. Efimov in the note ([1]), which is formulated in detail as follows.

Let

$$0 < \delta < \frac{2(4h\Delta^2 - 3)}{3(8h\Delta^2 - 3)}, \qquad M = \frac{4}{3}\left(\frac{2}{3\delta} - 1\right)\Delta. \tag{5}$$

If on some asymptotic there is an arc L for which

$$\int_L \sin\omega\, ds > M,$$

then for such an asymptotic, for some $\varepsilon > 0$, an ε-strip is impossible, $\varepsilon = \varepsilon(h, \Delta, \delta)$, subject to conditions (1), (2).

Hence, and from Lemma 1, Theorem A follows.

No. 4. We next present the main stages of the proof of assertion No. 3. Put

$$U = k^{-3/2}\left(\frac{\partial\omega}{\partial s_1} - \sin\omega\,\frac{\partial Q}{\partial s_2}\right).$$

Lemma 2. If

$$\int_L \sin\omega\, ds > M$$

on some arc L of the second family of asymptotics, then on L there is a point where

$$|U| > (2/3 - \delta)h\Delta.$$

Proof. This is obtained by contradiction with the aid of equation (1) of our note ([2]), as a consequence of which

$$\frac{\partial U}{\partial s_2} \geq \left\{h - \frac{3}{2\Delta}|U|\right\}\sin\omega.$$

Lemma 3. If on an asymptotic of the second family, somewhere

$$|U| > (2/3 - \delta)h\Delta,$$

then in some ε-strip of such an asymptotic there is a point where

$$|U| > \frac{3}{2}h\Delta; \qquad \varepsilon = \varepsilon(h, \Delta, \delta).$$

Proof. Suppose the condition of the lemma is satisfied at some point O and, consequently, on some segment OP of the first asymptotic. Through all points of OP draw asymptotics of the second family in the positive direction of this family, assuming $U > 0$ on OP (in the opposite case we change the orientations of all asymptotics; for the orientation conditions see the note ([2])). Denote by Ω the region occupied by the drawn asymptotics. Suppose $U \leq \frac{2}{3}h\Delta$ everywhere in Ω. We may assume $ek = 1$ on OP. From equation (III) of the note ([2]) we have

$$\frac{\partial}{\partial s_2}\{(ek)^3 U\} \geq h(ek)^3 \sin\omega; \tag{6}$$

as a consequence of (3),

$$\frac{\partial}{\partial s_2}\{(ek)^3 U\} \geq \frac{2}{3}h\Delta\,\frac{\partial (ek)^3}{\partial s_2}. \tag{7}$$

Integrating (6) and (7), we find:

$$(ek)^3 > \frac{2 - 3\delta}{2} > 0; \tag{8}$$

$$k^{-3/2}(ek)^3\frac{\partial\omega}{\partial s_1} > (ek)^3\left[\frac{2}{3}h\Delta - \frac{1}{2\Delta}\right] - \delta h\Delta. \tag{9}$$

From (9), taking into account (8) and (5), we have everywhere in Ω

$$k^{-3/2}\frac{\partial\omega}{\partial s_1} \geq N_1 = \text{const} > 0. \tag{10}$$

Let OT be an asymptotic of the second family, going in Ω from O; because $0 < \omega < \pi$, and because of (10), every asymptotic of the first family that goes into the domain Ω from any point of OT intersects every asymptotic of the second family passing in Ω. Therefore all points of Ω can be uniquely determined by coordinates (u, v), where u and v are single-valued coordinates on the segment OP and on the ray OT; in particular, $\omega = \omega(u, v)$.

From (10),

$$\frac{\partial\omega}{\partial u} \geq (ek)N_1. \tag{11}$$

Let O_1P_1 $(u_0 \leq u \leq u_1)$ be some segment lying strictly inside OP; let O_1T_1 be an asymptotic of the second family going in Ω from O_1; and let Ω_1 be the domain $(u_0 \leq u \leq u_1,\ v \geq 0)$. From (11) and (8) we have in Ω_1

$$\sin\omega \geq m = \text{const} > 0. \tag{12}$$

If s_1 is the length of the arc cut off by Ω_1 on some asymptotic of the first family, then

$$s_1 = \int_{u_0}^{u_1} e(u, v)\, du < \frac{\pi}{N_1}. \tag{13}$$

We may assume that $gk^{1/2} = 1$ on O_1T_1; then from (3) and (13) we have in Ω_1

$$e^{-\pi/\Delta N_1} \leq gk^{1/2} \leq e^{+\pi/\Delta N_1}. \tag{14}$$

Let $E_v^*(u)$ be the set of points $u \in [u_0, u_1]$ at which, for the given $v \geq 0$, one has $e(u, v)k(u, v) \geq \alpha/N_1$ $(\alpha = \text{const} > 0)$. From (11) we find $\operatorname{mes} E_v^*(u) \leq \pi/\alpha$. Consequently, for any $v \geq 0$, taking α sufficiently large, we have

$$\sqrt[8]{\frac{2-3\delta}{2}} < e(u, v)k(u, v) < \frac{\alpha}{N_1};$$

$$u \in E_v(u) = [u_0, u_1] - E_v^*(u), \qquad \operatorname{mes} E_v(u) \geq (u_1 - u_0) - \frac{\pi}{\alpha}.$$

From equation (III) of note [2], taking (12) into account,

$$\frac{\partial}{\partial s_2}\{(ek)^3U\} \geq hm(ek)^3k^{1/2}.$$

Hence, if $u \in E_v(u)$, then

$$\left| k^{-3/2}(ek)\sin\omega\,\frac{\partial Q}{\partial s_2} \right| \leq \frac{\alpha}{N_1}\frac{1}{2\Delta} = R = \text{const},$$

$$\frac{\partial\omega}{\partial u} \geq hm\left(\frac{N_1}{\alpha}\right)^2 \int_0^{s_2} (ek)^3k^{1/2}\,ds_2 - R.$$

Taking (14) and (8) into account, we find

$$\int_0^{s_2} (ek)^3k^{1/2}\,ds_2 = \int_0^{v} (ek)^3gk^{1/2}\,dv \geq e^{-\pi/\Delta N_1}\frac{2-3\delta}{2}\,v.$$

If $\alpha = 2\pi/(u_1-u_0)$, then $\operatorname{mes} E_v(u) \geq \frac{1}{2}(u_1-u_0)$, and $\partial\omega/\partial u \geq \lambda v - R$, $\lambda = \text{const} > 0$, $u \in E_v(u)$. Consequently, for sufficiently large v we shall have $\partial\omega/\partial u \geq 2\pi/(u_1-u_0)$, $u \in E_v(u)$, which contradicts the condition $0 < \omega < \pi$. Thus the assumption $U \leq \frac{2}{3}h\Delta$ must be rejected.

Lemma 4. If on an asymptotic of the second family, somewhere $|U| > {}^2/_3 h\Delta$, then for such an asymptotic, for some $\varepsilon > 0$, an ε-strip satisfying conditions (1), (2) is impossible; $\varepsilon = \varepsilon(h,\Delta)$.

Proof. We retain the preceding notation and suppose that $U > ({}^2/_3 + \delta)h\Delta$ on OP; here it is important only that $\delta = \text{const} > 0$. Analogously to (9) we find

$$k^{-3/2}(ek)^3\frac{\partial\omega}{\partial s_1} > (ek)^3\left[\frac{2}{3}h\Delta - \frac{1}{2\Delta}\right] + \delta h\Delta. \tag{15}$$

Hence $\partial\omega/\partial s_1 \geqslant N_1 = \text{const} > 0$, and the conclusions drawn in the preceding proof from (10) are preserved. Then from (15) we have

$$\frac{\partial\omega}{\partial u} \geqslant (ek)N_1 + \frac{1}{(ek)^2}N_2 \qquad (N_1, N_2 = \text{const} > 0). \tag{16}$$

From (16) follow (13) and (14). From (16) it also follows that

$$\sqrt{\frac{N_2}{\alpha}} < e(u,v)k(u,v) < \frac{\alpha}{N_1},$$

$u \in E_v(u)$, where $E_v(u)$ is a certain subset of the segment $[u_0, u_1]$, whose measure is
$\geqslant (u_1 - u_0) - 2\pi/\alpha$ (α is a sufficiently large number > 0). After this the proof is completed, in general, analogously to the proof of Lemma 3 (but with the use of Fubini' s theorem for the rectangle $u_0 \leqslant u \leqslant u_1,\ 0 \leqslant v \leqslant \text{const}$).

Moscow State University
named after M. V. Lomonosov

Received
25 X 1960

REFERENCES

1. N. V. Efimov, DAN, **136**, No. 6 (1961).
2. N. V. Efimov, E. G. Poznyak, DAN, **137**, No. 1 (1961).

Note: Figure translations are in progress. See original paper for figures.