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Abstract

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SOLUTION OF A SYSTEM OF QUASILINEAR EQUATIONS HAVING DIVERGENT FORM, UNDER PERIODIC BOUNDARY CONDITIONS

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In this note we prove existence and uniqueness theorems for solutions of certain classes of systems of quasilinear equations whose principal terms are written in divergent form, under prescribed periodic boundary conditions. It is assumed here that certain forms are positive; this requirement generalizes the condition of strong ellipticity for linear systems of equations (^{1}) and the regularity condition for variational problems. The case of systems with "weakly"nonlinear coefficients was considered in (^{2}).

1. Let a system of elliptic equations of the form* be given

$$\sum_{|\alpha|\leq m,\ |\beta|\leq m} (-1)^{|\alpha|} D_\alpha A_\alpha(x, u, D_\beta u) + B(x, u, D_\beta u) = h(x), \tag{1}$$

where $x = (x_1, \dots, x_n)$, $D_\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$. The functions $u = (u^1, \dots, u^N)$, $h = (h^1, \dots, h^N)$, $A_\alpha(\ )$, $B(\ )$ take vector values in R^N, and $A_\alpha(x, 0, 0) = 0$, $B(x, 0, 0) = 0$. Thus, (1) is a system of N equations with N unknowns u^j, written in vector form. A_α, B are continuously differentiable functions of their arguments for $x \in G$, $G = (0 \leq x_i \leq 2\pi)$, $-\infty < u, D_\beta u < +\infty$. For simplicity we restrict ourselves here to periodic boundary conditions

$$u\Big|_{x_i=0} = u\Big|_{x_i=2\pi}, \qquad D_\alpha u\Big|_{x_i=0} = D_\alpha u\Big|_{x_i=2\pi} \quad (i = 1, \dots, n). \tag{2}$$

Thus, we seek a solution of system (1) on the n-dimensional torus G.

Our main assumption, which will be made more precise below, reduces to the positivity of the form

$$l_1(u) \equiv [A_\alpha(x, u, D_\beta u), D_\alpha u] + [B(\ ), u] \geq c^2 \|u\|^{1+\delta}, \tag{3}$$

where $\delta > 0$, $\| \ \|$ is the norm in L_2 or L_p, $[\ ,\ ]$ is the scalar product of vector functions; $u(x) \in P^{2m}$, i.e. $u(x)$ is any function from $C^{2m}(G)$ satisfying (2); summation is to be taken over repeated indices. Condition (3) has functional form. It is easy to give an algebraic condition sufficient for (3) to hold:

$$(A_\alpha(x, u, \xi_\beta), \xi_\alpha) + (B(x, u, \xi_\beta), u) \geq c^2 |u|^{1+\delta}, \tag{4}$$

where ξ_α are arbitrary N-dimensional vectors, $x \in G$, $-\infty < u^j < +\infty$. Obviously, condition (3) is less restrictive than (4). Using the embedding theorems of S. L. Sobolev (^{3}), it is easy to give other sufficient conditions ensuring the validity of (3).

2. **Construction of approximate solutions.** Let

$$u_\chi = e^{i(k_1 x_1 + \cdots + k_{nx} n)},$$

where $\chi = (k_1, \dots, k_n)$ is any integral vector.

The approximate solution is sought by the Galerkin method in the form $u_i = \sum_{|\chi| \leq i} C_\chi u_\chi$, where $|\chi| = |k_1| + \dots + |k_n|$. The unknown coefficients C_χ are determined from the nonlinear system of equations:

$$K(u_i, u_\chi) \equiv [A_\alpha(x, u_i, D_\beta u_i), D_\alpha u_\chi] + [B(\ ), u_\chi] = [h, u_\chi] \quad (|\chi| \leq i). \tag{5}$$

The existence of a solution of this system follows from the following lemma:

Lemma. Let a finite system of nonlinear equations be given

$$A(\bar{C}) = \bar{h}, \qquad \bar{C} = (C_1, \dots, C_l), \qquad \bar{h} = (h_1, \dots, h_l), \tag{6}$$

where $A(\bar{C}) = (A^1(\bar{C}), \dots, A^l(\bar{C}))$ is a continuous vector function, $-\infty < C_j < +\infty$. If, for sufficiently large $|\bar{C}|$, the inequality

$$(A(\bar{C}), \bar{C}) \geq \gamma^2 |\bar{C}|^{1+\delta} \tag{7}$$

holds, then system (6) has at least one solution.

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* For simplicity of notation we assume that system (1) has the same order $2m$ with respect to all components u^j. Everything that follows remains valid, with obvious changes, if the order of (1) with respect to u^j is equal to $2m_j$, where the m_j may be different.

Indeed, on the sphere $|\bar{C}| = R > 1$, where $R > \left(\frac{1}{\gamma_1^2}|\bar{h}|\right)^{1/\delta_1}$, $\delta_1 = \min(\delta, 1)$, $\gamma_1^2 = \min(\gamma^2, 1)$,

$$(tA(\bar{C}) + (1-t)\bar{C} - \bar{h}, \bar{C}) \geq \gamma_1^2 |\bar{C}|^{1+\delta_1} - |\bar{h}| \cdot |\bar{C}| > 0. \tag{8}$$

Consequently, the vector field $A(\bar{C}) - \bar{h}$ on the sphere $|\bar{C}| = R$ is homotopic to the field $E\bar{C} - \bar{h}$, and in the process of deforming one field into the other we do not pass through the zero vector. It follows that the rotation of the field $A(\bar{C}) - \bar{h}$ on the sphere $|\bar{C}| = R$ is the same as the rotation of the field $E\bar{C} - \bar{h}$ (4). But the latter is equal to 1, and hence the former is equal to 1; therefore, for some $\bar{C}$ in the ball $|\bar{C}| < R$, $A\bar{C} - \bar{h} = 0$.

Applying this lemma to system (5), which also has the form (6) and which, according to (3), satisfies condition (7), we see that there exists at least one solution $\{C_\chi\}$ of system (5).

3. We shall prove the possibility of passage to the limit and the existence theorem for the solution of problem (1), (2) under two new assumptions, which, as a rule, already imply condition (3).

Condition R. For any s, $1 \leq s \leq n$,

$$\begin{aligned} l_2(u) \equiv \left[\frac{\partial}{\partial x_s} A_\alpha(x, u, D_\beta u), D_\alpha \frac{\partial u}{\partial x_s}\right] + \left[\frac{\partial}{\partial x_s} B(\ ), \frac{\partial u}{\partial x_s}\right] \geq \\ \geq \gamma^2 \sum_j \sum_{|\alpha|=m} \left[|D_\alpha u^j|^{r_j} D_\alpha \frac{\partial u^j}{\partial x_s}, D_\alpha \frac{\partial u^j}{\partial x_s}\right] + \ldots - f_1(l_1(u)), \end{aligned} \tag{9}$$

where u is any function from P^{2m}, the ellipsis denotes an analogous sum over $|\alpha| < m$, or it is absent; $r_j \geq 0$. Here and below f_i denotes some monotone function.

Condition P. For some $p > 1$, f_2 and f_3,

$$l_3(u) \equiv \sum_{|\alpha| \leq m} \|A_\alpha(x, u, D_\beta u)\|_p + \|B(\ )\|_p \leq f_2(l_1(u)) + f_3(l_2(u)), \tag{10}$$

where $\|\ \|_p = \|\ \|_{L_p}$, and for $|\alpha| = m$ and some $p_1 > 1$

$$\|D_\alpha u\|_{p_1} \leq f_4(l_1(u)) + f_5(l_2(u)). \tag{10'}$$

Obviously, in a number of cases one may, for greater precision, replace p in (10) by p_α, $p_\alpha > 1$. It is easy to give sufficient algebraic conditions for the fulfillment of inequalities (9) and (10). Below we present some of them.

Theorem 1. If conditions R, P, and (3) are satisfied, then there exists at least one solution $u(x)$ of problem (1), (2) for every function $h(x) \in \mathfrak{H}^{(2,5,6)}$. Moreover, the norms appearing on the left in (10) and (10′) are certainly finite, and for any function $v \in W_q^{(m)}(G)$, $1/q + 1/p = 1$, the relation holds-

solution (see notation (5))

$$K(u, v) = (h, v). \tag{11}$$

For brevity, we shall carry out the proof for the particular case (the general case is treated analogously) when in (1) $A_\beta = A_\alpha(D_\beta u)$ ($|\alpha| = m$, $|\beta| = m$) are homogeneous polynomials of order $2l + 1$ with respect to $D_\beta u^j$ ($j = 1, \dots, N$), $A_\alpha \equiv 0$ for $|\alpha| < m$, $B \equiv \omega^2 u$ (in the case of boundary conditions of the first boundary-value problem one may take $B \equiv 0$). We note that in this particular case the following is a sufficient condition for the fulfillment of conditions R, P, and (3):

$$\left(\frac{\partial A_\alpha(\zeta_\beta)}{\partial \zeta_{\beta_1}} \eta_{\beta_1}, \eta_\alpha \right) \geq \gamma^2 \sum_{j,\ |\beta| = m} |\zeta_\beta^j|^{2l} |\eta_\beta^j|^2 \tag{12}$$

for any vectors $\zeta_\beta = (\zeta_\beta^1, \dots, \zeta_\beta^N)$, $\eta_\beta = (\eta_\beta^1, \dots, \eta_\beta^N)$; $\partial A_\alpha / \partial \zeta_{\beta_1}$ for fixed α and β_1 is a matrix of order N. In the case (12), condition R is satisfied with $r_j = 2l$, $f_1 \equiv 0$. Substituting the vector $t\eta_\beta$ for ζ_β in (12) and integrating both sides with respect to t from 0 to 1, we obtain

$$\sum_\alpha (A_\alpha(\eta_\beta), \eta_\alpha) \geq \gamma_1^2 \sum_\beta |\eta_\beta|^{2l+2}. \tag{13}$$

Hence, by the choice $B = \omega^2 u$, (3) is satisfied with $\delta = 1$. Condition P, by virtue of (13), is also satisfied for $p = (2l + 2)/(2l + 1)$, $f_2 = Ct^{1/p} + Ct^{1/2}$, $f_3 \equiv 0$, $p_1 = 2l + 2$, $f_4(t) = t^{1/p_1}$, $f_5 \equiv 0$. The further exposition will be given for the case (12).

Multiply both sides of (5) by C_x and sum over x, $|x| \leq i$. We obtain

$$l_1(u_i) \equiv K(u_i, u_i) = [h, u_i], \tag{14}$$

and hence, by virtue of (3), we find that $l_1(u_i) \leq \|h\|^{1+1/\delta} = M$. By virtue of (13) (or condition P in the general case), it follows from this that

$$\|u_i\|^2 + \sum \|D_\alpha u_i\|_{2l+2}^{2l+2} \leq C l_1(u_i) \leq M_1. \tag{14'}$$

Next, since u_x are exponentials, $\partial u_x/\partial x_s = ik_{su}x$. Multiply both sides of (5) by $-k_s^2$, replace this multiplier for u_x by the operator $\partial^2/\partial x_s^2$, and transfer $\partial/\partial x_s$ to the first factor. We obtain (in the case under consideration)

$$\left[\frac{\partial A_\alpha(D_\beta u_i)}{\partial x_s},\, D_\alpha \frac{\partial u_x}{\partial x_s}\right] + \omega^2 \left[\frac{\partial u_i}{\partial x_s}, \frac{\partial u_x}{\partial x_s}\right] = -\left[h, \frac{\partial^2 u_x}{\partial x_s^2}\right]. \qquad (15)$$

Multiplying, as above, both sides of (15) by C_x, summing over x, $|x| \leq i$, and over s, and using (12), we derive the estimate ([ ] denotes the integral)

$$-\sum_s \left[h, \frac{\partial^2 u_i}{\partial x_s^2}\right] = \left[\frac{\partial A_\alpha(D_\beta u_i)}{\partial D_{\beta_1} u} D_{\beta_1} \frac{\partial u_i}{\partial x_s},\, D_\alpha \frac{\partial u_i}{\partial x_s}\right] + \omega^2 \sum_s \left\| \frac{\partial u_i}{\partial x_s} \right\|^2 \equiv$$

$$\equiv l_2(u_i) \geq \gamma^2 \sum_j \left[|D_\beta u_i^j|^{2l} \sum_s \left| D_\beta \frac{\partial u_i^j}{\partial x_s} \right|^2 \right] + \omega^2 \sum_s \left\| \frac{\partial u_i}{\partial x_s} \right\|^2. \qquad (16)$$

Since

$$\left| \left[h, \frac{\partial^2 u_i}{\partial x_s^2} \right] \right| = \left| - \left[\frac{\partial h}{\partial x_s}, \frac{\partial u_i}{\partial x_s} \right] \right| \leq C \left\| \frac{\partial h}{\partial x_s} \right\|^2 + \varepsilon \left\| \frac{\partial u_i}{\partial x_s} \right\|^2$$

and

$$\frac{\partial}{\partial x_s} (D_\beta u_i^j)^{l+1} = (l+1)(D_\beta u_i^j)^l \cdot D_\beta \frac{\partial u_i^j}{\partial x_s},$$

from (16) we obtain

$$\|(D_\beta u_i^j)^{l+1}\|_{W_2^{(1)}} \leq C_1,$$

or, by virtue of (13), (16),

$$\|(D_\beta u_i^j)^{l+1+z}\|_{W_2^{(1)}} \leq C_1 \qquad (z < 0 \text{ and } z \text{ small}). \qquad (17)$$

By virtue of (17) and (14′), from $\{u_i\}$ one can choose such a subsequence-

ness, which we shall also denote by $\{u_i\}$, such that: 1) $(D_\alpha u_i)^{l+1+z}$ converges almost everywhere to $(D_\alpha u)^{l+1+z}$, and hence $D_\alpha u_i$ converges almost everywhere to $D_\alpha u$ for any $|\alpha| \leq m$, where for $l = 2l_1$ it suffices to take $z = 0$, while for $l = 2l_1 - 1$ one takes $z = q/r$, where q and r are odd; 2) $(D_{\alpha_1} u_i^1)^{n_1} \cdots (D_{\alpha_N} u_i^N)^{n_N}$ converges weakly to $(D_{\alpha_1} u^1)^{n_1} \cdots (D_{\alpha_N} u^N)^{n_N}$ for any $n_1 + \cdots + n_N \leq 2l + 1$. (In the general case we ensure the convergence of $A_\alpha(x, u_i, D_\beta u_i)$, $B(x, u_i, \ldots)$ to

$A_\alpha(x, u, D_\beta u)$, $B(x, u, ...)$ on the basis of the estimates (10), (10′), (3). Property 1) follows from the complete continuity of the embedding operator: $W_2^{(1)} \to L_2$; 2) follows from estimate (14′). It is important that, thanks to the almost-everywhere convergence of $D_\alpha u_i$ to $D_\alpha u$, the powers of $D_\alpha u_i$ converge to the powers of one and the same function $D_\alpha u$, with u_i converging to u, and $D_\alpha u$ being a derivative of u.

For fixed χ we pass in (5) to the limit as $i \to \infty$, obtaining the same relation with u_i replaced by u. Since by linear combinations of u_χ one can approximate any function $v \in W_q^{(m)}(G)$ (in the case under consideration $p = (2l+2)/(2l+1)$, $q = 2l + 2$), the function u is a solution of problem (1), (2) in the sense of (11), i.e. it has derivatives up to order m. The theorem is proved.

4. **Theorem 2.** *If for some $p \leq 1$ the estimate*

$$\left\|\partial A_\alpha(x, w, D_\beta w)/\partial x_s\right\|_p + \|\partial B(\ )/\partial x_s\|_p \leq f_6(l_2(u)) + f_7(l_1(u)) \qquad (18)$$

holds for all $w \in P^{2m}$, then the constructed solution u has finite norms of the derivatives indicated on the left.

The proof follows from the fact that (18) is satisfied by the constructed approximations u_i, for which $l_1(u_i)$ and $l_2(u_i)$ are uniformly bounded in i.

For systems of second order ($m = 1$), in the case where the conditions of Theorems 1 and 2 are fulfilled, it follows from this that the constructed solution u satisfies equation (1) almost everywhere in G, with $\partial A_\alpha(x, u, D_\beta u)/\partial x_\alpha \in L_p$.

Let us note that in the polynomial example $A_\alpha(D_\beta u)$ considered above, (18) will be fulfilled if, instead of (12), one requires the more stringent condition

$$\left(\frac{\partial A_\alpha(\xi_\beta)}{\partial \xi_{\beta_1}} \eta_{\beta_1}, \eta_\alpha\right) \geq \gamma^2 \left(\sum_{j,\ |\beta|=2l} |\xi_\beta^j|^{2l}\right) \left(\sum_{j,\ |\beta_1|=2l} |\eta_{\beta_1}^j|^2\right). \qquad (19)$$

(One second-order equation of the form (1) under conditions of type (19) has been studied by other methods in a number of works (see, for example, [7–10].)

Theorem 3 (uniqueness). *If condition (12) is fulfilled in the polynomial case under consideration, or a condition generalizing (12) is fulfilled in the general case of equation (1), then the constructed solution u is unique in the class of functions with finite forms $l_1(u)$ and $l_2(u)$.*

Let u_1 and u_2 be two solutions of problem (1), (2) in the sense of (11). Subtracting term by term the relations (11) for $u = u_1$ and $u = u_2$, we obtain, in the simplest case (12),

$$[A_\alpha(D_\beta u_1) - A_\alpha(D_\beta u_2), D_\alpha v] + \omega^2[u_1 - u_2, v] = 0. \qquad (20)$$

Since $A_\alpha(D_\beta u_{1,2}) \in L_p$, where $p = (2l+2)/(2l+1)$, one may take $v = u_1 - u_2$ in (20). Introduce the auxiliary function

$$\varphi(t) = [A_\alpha((1-t)D_\beta u_2 + tD_\beta u_1) - A_\alpha(D_\beta u_2), D_\alpha(u_1 - u_2)] + \omega^2[...].$$

Obviously (by (20)), $\varphi(0) = 0$, $\varphi(1) = 0$. Hence $\varphi'(c) = 0$, $0 < c < 1$, whence we infer that u_1 coincides almost everywhere with u_2.

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Note: Figure translations are in progress. See original paper for figures.