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Abstract

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A. A. Dezin

Various Types of Invariant Systems of the Simplest Structure

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In the papers ([1,2]), boundary-value problems were considered for invariant elliptic and hyperbolic systems, i.e., for systems whose left-hand side admits a representation by means of the exterior differentiation operator d and the operator δ metrically adjoint to d ([3]). The operators d and δ, considered in a domain of Euclidean space, are simply differential operators with constant coefficients ([1]). But in the study of boundary-value problems their invariant character, i.e., the possibility of specifying them on an arbitrary Riemannian manifold, proves to be very essential.

The present paper aims to show that the stock of invariant systems admitting study by means of the methods developed is very broad and is by no means exhausted by the "strictly elliptic" and "strictly hyperbolic" cases considered in ([1,2,9]).

1. Parabolic equations of the type of the heat-conduction equations and "semi-decayed" parabolic equations

Among the simplest first-order systems with two unknown functions of two independent variables, besides the systems

$$D_1 {}^0u + D_2 {}^1u = {}^1f; \qquad -D_1 {}^1u \pm D_2 {}^0u = {}^0f; \qquad D_i \equiv \frac{\partial}{\partial x^i}$$

(where the plus sign corresponds to the elliptic case and the minus sign to the hyperbolic case, whose multidimensional analogues were considered in ([1,2])), one encounters the systems

$$\begin{aligned} D_1 {}^0u - D_2 {}^1u &= {}^1f, \\ -D_1 {}^1u + {}^0u &= {}^0f, \end{aligned} \qquad \begin{aligned} D_1 {}^0u - D_2 {}^1u &= {}^1f, \\ -D_1 {}^1u &= {}^0f, \end{aligned} \tag{1}$$

where in the first every smooth solution of the homogeneous system satisfies the heat-conduction equation, while the second ("semi-decayed") has the following characteristic feature: although unique solvability (under the corresponding boundary conditions) of the written equations is trivial, the usual energy inequality (estimating the L_2-norm of the solution in terms of the L_2-norm of the right-hand side) cannot be established for it ([4]).

The analogue of the first of the systems (1), for arbitrary n (the number of independent variables), is the system (n even)

$$
\begin{array}{rr}
d\,{}^0u + \delta\,{}^2u - \partial\,{}^1u = {}^1f, & \delta\,{}^1u + {}^0u = {}^0f, \\
d\,{}^2u + \delta\,{}^4u - \partial\,{}^3u = {}^3f, & d\,{}^1u + \delta\,{}^3u + {}^2u = {}^2f, \\
\dots\ \dots\ \dots\ \dots\ \dots & \dots\ \dots\ \dots\ \dots\ \dots \\
d\,{}^{n-4}u + \delta\,{}^{n-2}u - \partial\,{}^{n-3}u = {}^{n-3}f, & d\,{}^{n-3}u + \delta\,{}^{n-1}u + {}^{n-2}u = {}^{n-2}f, \\
d\,{}^{n-2}u - \partial\,{}^{n-1}u = {}^{n-1}f. &
\end{array}
\qquad \text{(P)}
$$

where, as in (2), $\partial \equiv \partial/\partial x^n$, while the operators d, δ act with respect to the variables $x^1, \dots, x^{n-1}$ on the covariant u (a system of functions) depending on x^n as on a parameter. It is not difficult to verify that every smooth solution of the homogeneous system (P) satisfies the equation $(d\delta + \delta d)u + \partial u = 0$, i.e. each component of the covariant u satisfies the heat equation.

It is interesting to note that the systems (P) and the systems (T) presented in (2) reveal a connection between the elliptic and parabolic or hyperbolic cases for first-order systems. It is known that if one simply appends derivatives with respect to "time" to the Cauchy—Riemann system, one obtains a "bad" system, belonging to none of the classical types. Meanwhile, if by K we denote the elliptic operator of the left-hand side of system (K) in (1), acting in $(n-1)$-dimensional space, and by K^* the formally adjoint operator, and denote by $\dot{u}, \ddot{u}$ the totalities of all covariants of odd or even degree, respectively (still considered in $(n-1)$-dimensional space and depending on x^n as on a parameter), then the system (P) can be written briefly in the form

$$K\ddot{u} - \partial\dot{u} = \dot{f}, \qquad K^*\dot{u} + \ddot{u} = \ddot{f}, \qquad \text{(P)}$$

and the system (T) from (2) in the form

$$K\ddot{u} - \partial\dot{u} = f, \qquad K^*\dot{u} + \partial\ddot{u} = \ddot{f}. \qquad \text{(T)}$$

The system that is a generalization of the second of the systems (1), in the notation introduced, has the form

$$K\ddot{u} - \partial\dot{u} = \dot{f}, \qquad K^*\dot{u} = \ddot{f}. \qquad (\hat{P})$$

In addition, one may consider systems obtained from (P) by replacing some of the $\partial\dot{u}$ by $\dot{u}$ in the left column or, conversely, some of the $\dot{u}$ by $\partial\dot{u}$ in the right column. In the first case we shall obtain "more elliptic" systems and, in the limit, replacing all $\partial\dot{u}$ by $\dot{u}$, we obtain (assuming that $\dot{u}$ does not depend on x^n) an inhomogeneous elliptic system embracing simultaneously all covariants of both even and odd degree. Replacing $\dot{u}$ by $\partial\dot{u}$ yields a "more hyperbolic" system, giving in the limit the system (T). Similarly, one may also consider systems occupying an intermediate position between the systems (P) and ($\hat{P}$).

For all the systems listed, correct boundary-value problems can be indicated in a domain of Euclidean space of the form $V \times e$, where V is a domain with smooth boundary and e is a unit interval. The proof of existence and uniqueness theorems for the generalized solution is carried out according to the same plan as in (2): V is completed to a closed Riemannian manifold M, unique solvability of the system is established on $M \times e$, and the transition to the boundary-value problem in the original domain is effected by considering subspaces of even or odd covariants.

To clarify what has been said, let us give the explicit form of the system (P_3) $(n = 3)$. It has the form:

$$\begin{aligned} D_1\overset{0}{u} + D_2\overset{2}{u} - D_3u_1 &= \overset{1}{f}_1, \qquad -D_1u_1 - D_2u_2 + \overset{0}{u} = \overset{0}{f}, \\ D_2\overset{0}{u} - D_1\overset{2}{u} - D_3u_2 &= \overset{1}{f}_2, \qquad D_1u_2 - D_2u_1 + \overset{2}{u} = \overset{2}{f}. \end{aligned} \tag{P_3}$$

If in the second pair of equations one deletes $\overset{0}{u}, \overset{2}{u}$, one obtains the equations $(\hat{P}_3)$; if one replaces $\overset{0}{u}, \overset{2}{u}$ by $D_3\overset{0}{u}, D_3\overset{2}{u}$, one obtains the hyperbolic system (T_3). If in the first pair of equations one replaces D_3u_1, D_3u_2 by u_1, u_2 and assumes that u does not depend on x^3, one obtains an elliptic system with characteristic determinant $(\xi_1^2 + \xi_2^2)^2$. If one deletes the terms with the derivative D_3 and the terms $\overset{0}{u}, \overset{2}{u}$ in the second pair, then a splitting into two formally conjugate Cauchy–Riemann systems (for $n = 2$ differing only in sign) occurs.

If V is a disk in the plane x^1, x^2 with boundary S, then for the system (P_3), for example, the boundary-value problem with boundary conditions is uniquely solvable for any right-hand side from L_2:

$$\overset{2}{u}\big|_S = -u_1 \cos\varphi + u_2 \sin\varphi\big|_S = 0; \tag{Γ_S}$$

$$u_1\big|_{x^3=0} = u_2\big|_{x^3=0} = 0, \tag{Γ}$$

where φ is the polar angle. When $\overset{0}{u}$ is replaced by $D_3\overset{0}{u}$, the condition $\overset{0}{u}\big|_{x^3=0} = 0$ is added; when D_3u_1 is replaced by u_1, the first of the conditions (Γ) is removed.

2. Systems with a structure independent of the dimension of the space. For all the systems considered (K), (T), (P), $(\hat{P})$, the following feature was characteristic: their properties remained unchanged for any n, but the form of the equations changed in passing from n to $n+1$, since, for example, the number of covariants entering the system changed. Meanwhile, since the operators d and δ are defined for any n, every system written with their aid in n-dimensional space automatically gives a certain new system in the $(n+1)$-dimensional case. True, the systems obtained in this way from the systems (K) will no longer have the property of unique solvability for any right-hand side from L_2 (cf. (5)); but for the "continued" systems (T), (P), $(\hat{P})$ it is not difficult to indicate correct boundary-value problems. Noting also that if n in the indicated systems is not identified with the dimension of the space, then the lowest degree of the covariants entering the system may be greater than zero, we may associate with the system (P), for example, the following systems (i odd):

$$
\begin{array}{ll}
d\overset{p}{\omega} + \delta\overset{p+2}{\omega} - \partial\overset{p+1}{\omega} = \overset{p+1}{f}, & \delta\overset{p+1}{\omega} + \overset{p}{\omega} = \overset{p}{f}, \\
\cdots\cdots\cdots\cdots & d\overset{p+1}{\omega} + \delta\overset{p+3}{\omega} + \overset{p+2}{\omega} = \overset{p+2}{f}, \\
d\overset{p+i-3}{\omega} + \delta\overset{p+i-1}{\omega} - \partial\overset{p+i-2}{\omega} = \overset{p+i-2}{f}, & \cdots\cdots\cdots\cdots \\
d\overset{p+i-1}{\omega} - \partial\overset{p+i}{\omega} = \overset{p+i}{f}; & d\overset{p+i-2}{\omega} + \delta\overset{p+i}{\omega} + \overset{p+i-1}{\omega} = \overset{p+i-1}{f},
\end{array}
\qquad (P_i^p)
$$

where i may take the values $1, \dots, n-p$, and p the values $0, 1, \dots, n-i$. Analogous "two-index variants" may be written for the systems (T) and $(\hat{P})$. As was noted, correct boundary-value problems can be indicated for all such systems.

3. Two classical examples. Since the differential operators of classical vector analysis are special examples of the operators d, δ, it is natural to ask what first-order linear systems of equations, occurring in various branches of physics and written in vector form, look like from the point of view of the considerations carried out above. As a partial answer to this question we give two examples.

The first is Maxwell' s equations. Written in the so-called four-dimensional form, i.e. for the skew-symmetric tensor of the electromagnetic of the field $F_{ik}^{(6)}$; in the case of a field in vacuum, in the absence of charges, they have the form (the constants have been set equal to 1)

$$d\overset{2}{F} = 0; \qquad \delta\overset{2}{F} = 0, \tag{2}$$

where the operator δ is computed in the Lorentz metric. The splitting carried out in (2) (when time is present) of the operators d and δ corresponds exactly to the passage from equations (2) to their usual three-dimensional form, which in invariant notation has the form

$$\partial \overset{1}{u} + * \, d\overset{1}{v} = 0, \qquad \partial \overset{1}{v} - * \, d\overset{1}{u} = 0,$$
$$d * \overset{1}{u} = 0; \qquad \delta \overset{1}{v} = 0. \tag{3}$$

In passing from (2) to (3) we have put $\overset{2}{F} = (\overset{1}{u}, \overset{1}{v})$ ([2]), and, in order to deal only with vectors or 1-covariants, $\overset{2}{u} = *\overset{1}{u}$ ([3]). The upper equations (3) give the usual relations for curls, and the lower ones—the vanishing of the divergences.

A second example is provided by the (linearized) equations of hydrodynamics for a rotating fluid ([7,8]). In our notation they have the form

$$\partial \overset{1}{v} - *(\overset{1}{v} \wedge \overset{1}{k}) + d\overset{0}{p} = \overset{1}{f}; \qquad \eta \, d\overset{0}{p} - \delta \overset{1}{v} = \overset{0}{f}, \tag{4}$$

where $\eta = 0$ in the case of an incompressible fluid and $\eta = 1$ for a compressible one; $\overset{1}{v}$ and $\overset{0}{p}$ are unknowns; $\overset{1}{k}$ is a prescribed vector not depending on time.

The methods developed for studying invariant systems prove to be very convenient in the consideration of boundary-value problems for equations (2)–(4) and for some of their generalizations.

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Note: Figure translations are in progress. See original paper for figures.