ON THE EXISTENCE OF PERIODIC SOLUTIONS IN CERTAIN NONLINEAR SYSTEMS

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Abstract

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MATHEMATICS

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Consider the system of differential equations

$$\frac{dx}{dt} = Ax + F(x,t), \tag{1}$$

where x and $F(x,t)$ are real n-dimensional vectors with components $x_1, x_2, \dots, x_n$ and $F_1(x,t), \dots, F_n(x,t)$, respectively; $A = \{a_{ij}\}$ is a real constant square matrix of order n. As to the function $F(x,t)$, it is assumed that it is continuous, has period 2π in the variable t, and satisfies the uniqueness condition for solutions for all x, t.

The purpose of the present note is to establish sufficient conditions for the existence of a 2π-periodic solution of system (1). M. A. Krasnosel' skii studied an analogous problem ([1,2]). The conditions given below for the existence of 2π-periodic solutions differ from those found by M. A. Krasnosel' skii in that, under these conditions, the matrix A may have imaginary eigenvalues.

Introduce the following notation. Let ξ be an n-dimensional vector with components $\xi_1, \dots, \xi_n$; let $B = \{b_{ij}\}$ be a square matrix of order n. Put

$$\|\xi\| = \sum_{i=1}^{n} |\xi_i|, \qquad \|B\| = \sum_{i,j=1}^{n} |b_{ij}|.$$

Theorem. *If the matrix A has no eigenvalues of the form ki (k a natural number or zero, i the imaginary unit), and if the function $F(x,t)$, for sufficiently large $\|x\|$, satisfies the inequality*

$$\|F(x,t)\| < L\|x\|, \tag{2}$$

where the positive constant L is sufficiently small, then system (1) has at least one 2π-periodic solution.

Proof. Denote by $x(t, x^0, t_0)$ the solution of system (1) with initial data $t_0, x^{(0)}$. From estimates (2) it follows ([3,4]) that all solutions of system (1) can be continued for all times from $-\infty$ to $+\infty$. Associate with the point $x^{(0)}$ the point

$x(2\pi, x^{(0)}, 0)$. Thus we obtain a topological transformation T of the space $\{x\}$ into itself. Along with this transformation, consider the vector transformation $v = T(x) - x$ with components $v_1, v_2, \dots, v_n$. It is clear that the fixed points of this transformation or, what is the same, the singular points of the field v, are the initial data of 2π-periodic solutions, and conversely.

Now consider the linear system

$$\frac{dx}{dt} = Ax. \tag{3}$$

Without loss of generality, we may assume that the matrix A has canonical structure. Define, for system (3), a transformation T_0 of the space $\{x\}$ into itself, analogous to the transformation T for system (1). Denote by $w = \{w_1, \dots, w_n\}$ the vector of the transformation T_0, i.e. $w = T_0(x) - x$. Let the matrix A have the following characteristic roots: $\lambda_1 \pm i\mu_1, \lambda_2 \pm i\mu_2, \dots, \lambda_s \pm i\mu_s, \varkappa_1, \dots, \varkappa_{n-2s}$, among which some may be equal. It is clear that

$$\frac{D(w_1, \dots, w_n)}{D(x_1, \dots, x_n)} = \begin{vmatrix} e^{2\pi\lambda_1} \cos 2\pi\mu_1 - 1 & -e^{2\pi\lambda_1} \sin 2\pi\mu_1 & 0 \dots 0 \\ e^{2\pi\lambda_1} \sin 2\pi\mu_1 & e^{2\pi\lambda_1} \cos 2\pi\mu_1 - 1 & 0 \dots 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots e^{2\pi\varkappa_{n-2s}} - 1 \end{vmatrix},$$

and by Laplace' s theorem we have

$$\frac{D(w_1, \dots, w_n)}{D(x_1, \dots, x_n)} = \prod_{m=1}^{s} \left(1 - 2e^{2\pi\lambda_m} \cos 2\pi\mu_m + e^{4\pi\lambda_m}\right) \prod_{m=1}^{n-2s} \left(e^{2\pi\varkappa_m} - 1\right).$$

By virtue of the condition of the theorem that in the case $\lambda_m = 0$ one has $\mu_m \neq k$ (k is a natural number or zero), we have

$$\frac{D(w_1, \dots, w_n)}{D(x_1, \dots, x_n)} \neq 0.$$

This proves (see, for example, ([5])) that the index of the point $x_1 = \dots = x_n = 0$ as a singular point of the field $w(x)$ is equal to ± 1. Moreover, from the inequality

$$D(w_1, \dots, w_n)/D(x_1, \dots, x_n) \neq 0$$

it follows that there exists a constant $K > 0$ such that

$$\|w(x)\| \geq K\|x\|. \tag{4}$$

Let us now estimate the difference $\|v(x^{(0)})-w(x^{(0)})\|$ for sufficiently large values of $\|x^{(0)}\|$. It is clear that

$$v(x^{(0)}) - w(x^{(0)}) = x(2\pi, x^{(0)}, 0) - y(2\pi, x^{(0)}, 0), \tag{5}$$

where $y(t, x^{(0)}, t_0)$ denotes the solution of system (3) with initial data $t_0, x^{(0)}$. Let $Y(t)$ be the solution of the matrix equation

$$dY/dt = AY \tag{6}$$

such that $Y(0) = I$.

It is known (see, for example, ([6])) that then the solution $x(t, x^{(0)}, 0)$ of system (1) satisfies the integral equation

$$x(t, x^{(0)}, 0) = y(t, x^{(0)}, 0) + \int_0^t Y(t-\tau)F(x,\tau)\,d\tau. \tag{7}$$

Put

$$c_1 = \sup_{0\le t\le 2\pi} \|y(t, x^{(0)}, 0)\|;$$

since $y(t, x^{(0)}, 0)$ is a solution of the linear system (3), there exists a positive constant M such that $c_1 \le M\|x^{(0)}\|$. Let, moreover,

$$c_2 = \sup_{0\le t\le 2\pi} \|Y(t)\|.$$

Then from equality (7) we obtain

$$\|x(t, x^{(0)}, 0)\| \le c_1 + c_2 \int_0^t \|F(x(\tau, x^{(0)}, 0), \tau)\|\,d\tau \quad \text{for } 0 \le t \le 2\pi.$$

Assuming that $\|x^{(0)}\|$ is sufficiently large, from condition (2) we obtain

$$\|x(t, x^{(0)}, 0)\| < c_1 + c_2 L \int_0^t \|x(\tau, x^{(0)}, 0)\|\,d\tau \quad \text{for } 0 \le t \le 2\pi. \tag{8}$$

Hence, by a well-known lemma (see, for example, ([6], [7])), we find

$$\|x(t, x^{(0)}, 0)\| < c_1 e^{c_2 L t} \quad (0 \le t \le 2\pi). \tag{9}$$

On the other hand, from equality (7) there follows the estimate

$$\|x(t, x^{(0)}, 0) - y(t, x^{(0)}, 0)\| \leq c_2 \int_0^t \|F(x, \tau)\| \, d\tau \quad (0 \leq t \leq 2\pi),$$

and hence, for sufficiently large $\|x^{(0)}\|$, taking into account (2), (9), we obtain

$$\|x(t, x^{(0)}, 0) - y(t, x^{(0)}, 0)\| < c_2 L c_1 \int_0^t e^{c_2 L t} \, dt = c_1 (e^{c_2 L t} - 1).$$

Hence and from (5) we conclude, in view of $c_1 \leq M\|x^{(0)}\|$,

$$\|v(x^{(0)}) - w(x^{(0)})\| < M(e^{2\pi c_2 L} - 1)\|x^{(0)}\| \tag{10}$$

for sufficiently large $\|x^{(0)}\|$.

Let now R be a sphere of sufficiently large radius with center at the origin. We shall take the constant L so small that the inequality

$$K \geq M(e^{2\pi c_2 L} - 1) \tag{11}$$

is satisfied.

Then from estimates (4) and (10) it follows that at no point of the sphere R are the vectors v and w oppositely directed, and, moreover, v does not vanish at any point of the sphere R. It is then clear that the indices of the sphere R in the fields w and v coincide. We have already proved that the index of the point $x_1 = \cdots = x_n = 0$, as a singular point of the field $w(x)$, is equal to ± 1. But the sphere R, by virtue of inequality (4), contains no singular points of the field w other than $x_1 = \cdots = x_n = 0$, and therefore its index in the field w is equal to ± 1. Consequently, the index of the sphere R in the field v is also equal to ± 1, and this means that the sphere R contains at least one singular point of the field v. Thus, system (1) has at least one 2π-periodic solution. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.