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Abstract

Full Text

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ON GENERAL SYSTEMS OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

(Presented by Academician A. N. Kolmogorov on 17 XII 1960)

In Hörmander's paper ⁽¹⁾, among other things, a somewhat unexpected result was obtained, namely that for any differential operator with constant coefficients in an arbitrary domain Ω of the space R^ν there exists a correctly posed boundary-value problem (correctness is understood here in the sense of M. I. Vishik, see ⁽⁴⁾). The significance of this theorem, which is essentially a pure existence theorem, should not, however, be overestimated, since even in the simplest cases it does not make it possible to indicate the boundary conditions that realize the correct problem.

When one passes to systems of differential equations with constant coefficients, Hörmander's theorem turns out to be false; this was first pointed out by A. A. Dezin ⁽³⁾. He constructed a system of equations for which correct solvability was ensured in the space W_2^1 , but no solvable extension in L_2 existed.

In the present note we give necessary and sufficient conditions for an operator-system a to be subordinate to an operator b . These conditions are described by means of determinants of certain matrices and, in the case where a is the identity operator, give a criterion for the existence of a correct problem for the operator b . The proof of the criterion is carried out by a new method, different from the energy-integral method usually used in such cases. At the same time, the above-mentioned Hörmander theorem can be derived as a special case.

We shall use the following notation: R^ν is the real ν -dimensional space of independent variables $x = (x^1, \dots, x^\nu)$; R_ν is the real space of elements $\xi = (\xi_1, \dots, \xi_\nu)$; $P(\xi) = \sum a^\alpha \xi_\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_k)$ is a sequence of indices between 1 and ν , is an arbitrary polynomial with complex (constant) coefficients a^α ; $\mathcal{P} = P(\mathcal{D})$, $D_j = \frac{1}{i} \frac{\partial}{\partial x^j}$, $\mathcal{D}_\alpha = \mathcal{D}_{\alpha_1} \dots \mathcal{D}_{\alpha_k}$ is the differential operator defined by it, acting on functions defined on R^ν ; if G is a domain in R^ν , then $C_0^\infty(G)$ is the set of complex-valued infinitely differentiable vector-functions $u(x) = (u_1(x), \dots, u_n(x))$ that vanish outside compact subsets of G ; $H(G)$ is the closure of $C_0^\infty(G)$ in the usual norm

$$\|u, H\| = \left(\sum_{j=1}^n \int_G |u_j(x)|^2 dx \right)^{1/2} = \left(\sum_{j=1}^n \|u_j, L_2\|^2 \right)^{1/2}; \quad (1)$$

\mathcal{H} is the space dual to $H(G)$ in the sense of the Fourier transform, i.e. $\hat{u}(\xi) \in \mathcal{H}$ if and only if

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_G u(x) e^{i\langle x, \xi \rangle} dx, \quad u(x) \in H(G);$$

$a = \|a_{ij}\|$ and $b = \|b_{ij}\|$ are square matrices of the corresponding orders.

Let $a = \|a_{ij}(\mathcal{D})\|$, $i, j = 1, \dots, n$, be a differential operator-system in which the $a_{ij}(\xi)$ are arbitrary polynomials. Denote by $\mathcal{A} = \|A_{ij}(\mathcal{D})\|$, $i, j = 1, \dots, n$, the operator whose elements are the algebraic cofactors of the elements of the operator a . Let $\Lambda = \|\delta_{ij}\Delta\|$, where $\Delta = \det \|a_{ij}\|$, and δ_{ij} is the Kronecker symbol. Then

$$a\mathcal{A} = \Lambda. \tag{2}$$

Let G be a bounded domain in R^{ν} .

Definition. We shall say that an operator b is subordinate to the operator a , and write this as $b < a$, if, with some constant γ ,

$$\|bu, H\| \leq \gamma \|au, H\|, \quad u \in C_0^\infty(G). \tag{3}$$

We shall give a description of all operators $b = \|b_{ij}(\mathcal{D})\|$ that are subordinate to the operator a . To formulate the result, following Hörmander, we introduce the function

$$\tilde{P}(\xi) = \left(\sum_{\alpha} |P^{(\alpha)}(\xi)|^2 \right)^{1/2},$$

where P is an arbitrary polynomial. Further, for given operators a and b define n^2 operators $S_{ij}(\mathcal{D})$, obtained from a by replacing its i -th row by the j -th row of b . Let $\Delta_{ij}(\mathcal{D}) = \det \|S_{ij}\|$.

Theorem 1. In order that the operator b be subordinate to the operator a , it is necessary and sufficient that, for every value $i, j = 1, \dots, n$,

$$\frac{|\Delta_{ij}(\xi)|}{\tilde{\Delta}(\xi)} \leq C \tag{4}$$

for all real ξ , where C is a constant.

Necessity. Let $b < a$. Then for all $u \in C_0^\infty(G)$,

$$\|bu, H\| \leq \gamma \|au, H\|.$$

Since for any $v \in C_0^\infty(G)$, $Av \in C_0^\infty(G)$, by virtue of (2) we have

$$\|bAv, H\| \leq \gamma \|\Lambda v, H\|. \quad (5)$$

Inequality (5) holds, in particular, for the vector $v^j = (0, \dots, \dots, v_j, \dots, 0)$, where v_j is an arbitrary smooth function equal to zero outside a compact subset of G . In this case, as is easily verified, (5) takes the form

$$\sum_{k=1}^n \|\Delta_{jk} v_j, L_2\| \leq \gamma \|\Delta v_j, L_2\|. \quad (5')$$

Hence, for each fixed k , $1 \leq k \leq n$,

$$\|\Delta_{jk} v_j, L_2\| \leq \gamma \|\Delta v_j, L_2\|.$$

By the arbitrariness of v_j it has been proved that the operator $\Delta_{jk} < \Delta$, $k = 1, \dots, n$, and since j can be chosen arbitrarily, $\Delta_{jk} < \Delta$ for all values of the indices. Inequality (4) now follows from Theorem 2.2 of paper ⁽¹⁾.

In proving the sufficiency of condition (4), the following basic lemma is used:

Lemma. Let $P(\xi) = P(\xi_1, \dots, \xi_\nu)$ be an arbitrary polynomial. Then there exists a set $\mathcal{M} \subset R_\nu$ having the following properties:

1°. $\mathcal{M} = \bigcup_1^n \mathcal{M}_k$, $n < \infty$, and for each of the sets \mathcal{M}_k there exists a direction $\vec{\eta}_k$ such that every vector y parallel to $\vec{\eta}_k$ intersects \mathcal{M}_k in a finite number of segments whose total length does not exceed a certain number l_k . (The number l_k depends only on the direction $\vec{\eta}_k$ and can be chosen arbitrarily small.)

2°. If $\mathfrak{M} = R_\nu \setminus \mathcal{M}$, then for all $\xi \in \mathfrak{M}$

$$\frac{|P(\xi)|}{\tilde{P}(\xi)} \geq C, \quad 1 > C > 0.$$

3°. There exists a constant $\gamma < 1$ such that for all $\hat{u}(\xi) \in \mathcal{H}$

$$\int_{\mathcal{M}} |\hat{u}(\xi)|^2 d\xi \leq \gamma \int_{R_\nu} |\hat{u}(\xi)|^2 d\xi. \quad (6)$$

Let us note that, in order to prove the lemma, it is enough to construct a set \mathcal{M} satisfying conditions 1° and 2°. Then, as follows from the results of [2], \mathcal{M} will automatically satisfy condition 3°, provided care is taken that the sum $L = \sum_{k=1}^n l_k$ is sufficiently small.

Let us now observe that inequality (3) means the continuity of the operator $T : Tau = bu$, $u \in C_0^\infty(G)$, and therefore, in order to prove this inequality, by

Banach's theorem it is enough to show that from $au_k, H\| \rightarrow 0$ it follows that $\|bu_k, H\| \rightarrow 0, k \rightarrow \infty, u_k \in C_0^\infty(G)$.

Thus, suppose that conditions (4) are satisfied. Suppose also that

$$au_k = f_k, \quad bu_k = g_k, \quad u_k \in C_0^\infty(G); \quad (7)$$

$$\|f_k, H\| \rightarrow 0, \quad k \rightarrow \infty. \quad (8)$$

Applying the operator \mathcal{A} to the first of equations (7), and the operator b to the resultant, and using the commutativity of the operators Λ and b , we find that

$$\Lambda g_k = \mathcal{B}f_k, \quad k = 1, 2, \dots, \quad (9)$$

where the operator $\mathcal{B} = \|\Delta_{ij}(\mathcal{D})\|$, or, in Fourier transforms,

$$\Lambda(\xi)\hat{g}_k(\xi) = \mathcal{B}(\xi)\hat{f}_k(\xi), \quad k = 1, 2, \dots. \quad (9')$$

Write out the i -th row of equality (9'). It has the form

$$\Delta(\xi)\hat{g}_{ki}(\xi) = \sum_{j=1}^n \Delta_{ij}\hat{f}_{kj}(\xi), \quad k = 1, 2, \dots. \quad (9'')$$

Divide both sides of (9'') by $\widetilde{\Delta}(\xi)$. Then from (8), by virtue of condition (4) and Parseval's equality, it follows that

$$\left\| \frac{\Delta(\xi)}{\widetilde{\Delta}(\xi)} \hat{g}_{ki}(\xi), \mathcal{H} \right\| \rightarrow 0, \quad k \rightarrow \infty.$$

In other words,

$$\int_{R_\nu} \frac{|\Delta(\xi)|^2}{\widetilde{\Delta}(\xi)^2} |\hat{g}_{ki}(\xi)|^2 d\xi \rightarrow 0, \quad k \rightarrow \infty. \quad (10)$$

We now construct, for the polynomial $\Delta(\xi)$, the corresponding set \mathcal{M} , satisfying conditions 1°–3° of the lemma, and let, as before, $\mathfrak{M} = R_\nu \setminus \mathcal{M}$. Then for $\xi \in \mathfrak{M}$

$$\frac{|\Delta(\xi)|}{\widetilde{\Delta}(\xi)} \geq C > 0, \quad (11)$$

whence we find

$$\int_{R_\nu} \frac{|\Delta(\xi)|^2}{\widetilde{\Delta}(\xi)^2} |\hat{g}_{ki}(\xi)|^2 d\xi > \int_{\mathfrak{M}} \frac{|\Delta(\xi)|^2}{\widetilde{\Delta}(\xi)^2} |\hat{g}_{ki}(\xi)|^2 d\xi \geq C \int_{\mathfrak{M}} |\hat{g}_{ki}(\xi)|^2 d\xi.$$

Condition (6), as is easy to see, entails the inequality

$$\int_{\mathfrak{M}} |\hat{g}_{ki}(\xi)|^2 d\xi \geq (1 - \gamma) \int_{R_\nu} |\hat{g}_{ki}(\xi)|^2 d\xi, \quad \gamma < 1,$$

and from (10), thus, it follows that $\|\hat{g}_k^i(\xi), \mathcal{H}\| \rightarrow 0, k \rightarrow \infty$. Since this is true for arbitrary $i = 1, \dots, n$, it is proved that if $\|au_k, H\| \rightarrow 0$, then $\|bu_k, H\| \rightarrow 0, k \rightarrow \infty$. The proof of Theorem 1 is complete.

Corollary. Let $a = \|a_{ij}(\mathcal{D})\|$ be an arbitrary differential operator with constant coefficients, and let $\lambda_{ij}(\mathcal{D})$ be the corresponding principal minors of the matrix a . In order that the operator a have at least one well-posed problem, it is necessary and sufficient that, for all real ξ ,

$$\frac{|\lambda_{ij}(\xi)|}{\widetilde{\Delta}(\xi)} \leq C, \quad i, j = 1, \dots, n. \quad (12)$$

Let a^* denote the operator formally adjoint to a . Then (see ⁽¹⁾, Theorem 1.2), for well-posed problems to exist for the operator a it is necessary and sufficient that, for all $u \in C_0^\infty(G)$,

$$\|au, H\| \geq c\|u, H\|, \quad \|a^*u, H\| \geq c^*\|u, H\|.$$

The operator b occurring in Theorem 1 is in this case simply the identity operator E , while the operators $\Delta_{ij}(\mathcal{D})$, evidently, coincide with $\lambda_{ij}(\mathcal{D})$, and it remains to note that, according to Theorem 1, both inequalities hold or fail to hold simultaneously.

Remark 1. The method of proving the sufficiency of condition (4), without any changes (except for specifically matrix details), carries over to the case of two differential polynomials a and b . Indeed, we wish to prove that (see ⁽¹⁾), if

$$\frac{|b(\xi)|}{\bar{a}(\xi)} < C, \quad (4')$$

then $\|au, L_2\| \geq \mu\|bu, L_2\|, u \in C_0^\infty(G), \mu = \text{const} > 0$. We shall prove that if

$$au_n = f_n, \quad bu_n = g_n, \quad u_n \in C_0^\infty(G) \quad (13)$$

and $\|f_n, L_2\| \rightarrow 0$, then $\|g_n, L_2\| \rightarrow 0, n \rightarrow \infty$. From (13) it is clear that $ag_n = bf_n$, or, in Fourier transforms,

$$a(\xi)\hat{g}_n(\xi) = b(\xi)\hat{f}_n(\xi). \quad (14)$$

Divide (14) by $\tilde{a}(\xi)$. Then, from the condition of the theorem and from $\|\hat{f}_n(\xi), \mathcal{H}\| \rightarrow 0, n \rightarrow \infty$, it follows that

$$\left\| \frac{a(\xi)}{\tilde{a}(\xi)} \hat{g}_n(\xi), \mathcal{H} \right\| \rightarrow 0,$$

and, applying the lemma, we find that

$$\|\hat{g}_n(\xi), \mathcal{H}\| = \|g_n, L_2\| \rightarrow 0.$$

Remark 2. We wish to draw special attention to the fact that in the case of systems of differential equations the notions of **strength** and **subordination** of two operators do not coincide. More precisely, if $b < a$, then a is stronger than b . The converse, however, is false. Indeed, for any operator a the identity operator E is weaker than a , but, as the corollary to Theorem 1 shows, not always $E < a$.

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Note: Figure translations are in progress. See original paper for figures.

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