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# MATHEMATICS

O. A. OLEINIK

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**Abstract**

**Full Text**

MATHEMATICS

O. A. OLEINIK

## ON QUASILINEAR PARABOLIC EQUATIONS WITH MANY INDEPENDENT VARIABLES

*(Presented by Academician I. G. Petrovskii, 14 XII 1960)*

Below we give a proof of the existence and uniqueness of a classical solution of the Cauchy problem and of the first boundary-value problem for one class of quasilinear parabolic equations of second order, and indicate an approximate method for solving them. The existence theorems for solutions of these problems will be obtained on the basis of the well-known Leray–Schauder theorem <sup>(1)</sup>. In deriving a priori estimates for the solutions we make substantial use of the work of J. Nash <sup>(2)</sup> and of the method for estimating the maxima of the moduli of derivatives proposed in <sup>(3)</sup>.

We shall consider an equation of the form

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(t, x, u) \frac{\partial u}{\partial x_j} \right) + a(t, x, u, u_x) - \frac{\partial u}{\partial t} = 0, \quad (1)$$

where

$$0 < \lambda_1(M) \leq \sum_{i,j=1}^n a_{ij} \alpha_i \alpha_j \leq \lambda_2(M),$$

if  $|\alpha| = 1$ , for all  $x$ ,  $0 \leq t \leq T$ ,  $|u| \leq M$ ; the coefficients  $a_{ij}$ ,  $a$  are sufficiently smooth functions;  $a(t, x, u, p)$ ,  $a_{ij}(t, x, u)$  and their derivatives are bounded for  $|u| \leq M$  ( $M$  is an arbitrary number) and all values of  $p$ ,  $x$  and  $t \geq 0$ ;  $a_{ij} = a_{ji}$ ,  $a_u < c$ . Obviously, one may assume that  $c < 0$ .

**1. The Cauchy problem.** In the layer  $Q\{T \geq t \geq 0\}$  we seek a bounded solution of equation (1) satisfying the condition

$$u(0, x) = u_0(x), \quad \text{where } |u_0(x)| \leq M_0. \quad (2)$$

In what follows, by  $M_l$  ( $l = 1, 2, \dots$ ) we shall denote positive constants depending only on  $c, T, M_0$ , the maxima of the moduli of the derivatives of  $u_0(x)$ , the

maxima of the moduli of the functions  $a_{ij}$ ,  $a$  and of their derivatives for  $0 \leq t \leq T$ ,  $|u| \leq M_0 + A/|c| = M_1$ , where  $A = \max |a(t, x, 0, u_x)|$ . We establish a number of a priori estimates for the solution  $u(t, x)$  of the problem (1), (2), assuming that  $u, u_{x_i}, u_t, u_{x_i x_j}$  are bounded in  $Q$  and satisfy a Hölder condition.

**Lemma 1.** For the solution  $u(t, x)$  of the problem (1), (2) in  $Q$ , the inequalities

$$|u(t, x)| \leq M_1, \quad |u(t, x) - u_0(x)| \leq M_2 t \quad (3)$$

hold.

The estimates (3) are a consequence of the maximum principle.

We shall assume that, for  $(t, x) \in Q$  and  $|u| \leq M_1$ ,  $a_{ij} \in C^{3+\alpha}$ ,  $a \in C^{2+\alpha}$ ,  $0 < \alpha \leq 1$ , and  $u_0(x) \in C^3$ . Using (6), it is easy to show that, under these assumptions on the smoothness of  $a_{ij}$ ,  $a$  and  $u(t, x)$ , equation (1) can be differentiated twice with respect to  $x_i$  inside  $Q$ .

**Lemma 2.** In the domain  $Q_1\{0 \leq t \leq M_3, -\infty < x < +\infty\}$

$$|u_{x_i}| \leq M_4, \quad \text{where } i = 1, \dots, n. \quad (4)$$

For the proof of this lemma we use a device from the paper (3). In equation (1) we make the substitution  $u = \varphi(v)$ , where  $\varphi' > 0$  and  $(\varphi''/\varphi')' < 0$ . We differentiate the resulting equation with respect to  $x_k$ , multiply it by  $p_k = \partial v / \partial x_k$ , and sum over all  $k$ . It follows from this equation that at the point of an interior maximum of the function  $a_\varepsilon(x - x_0)p^2$ , where  $a_\varepsilon(x)$  is a sufficiently smooth function,  $a_\varepsilon(x) \equiv 0$  for  $|x| \geq 2\varepsilon$ ,  $a_\varepsilon(x) \equiv 1$  for  $|x| \leq \varepsilon$ , and  $p^2 \equiv |\text{grad } v|^2$ , the inequality

$$a_\varepsilon p^4 [(\varphi''/\varphi')' + M_5|\varphi'|^2 + M_6|\varphi''| + 1/4] \geq -M_7 p^2. \quad (5)$$

is satisfied. Choose  $\varepsilon$  and  $M_3$  so that in the domain  $D_0\{|x - x_0| \leq 2\varepsilon, 0 \leq t \leq M_3\}$  the oscillation of  $u(t, x)$ , equal to  $\max u(t, x) - \min u(t, x) = M_8$ , is so small that the function

$$\varphi(v) = \min_{D_0} u - 2M_8 + 3eM_8 \int_0^v e^{-s^2} ds \quad (6)$$

satisfies the inequality

$$-(\varphi''/\varphi')' > 1/2 + M_5|\varphi'|^2 + M_6|\varphi''|. \quad (7)$$

With this choice of  $\varphi(v)$ , it follows from inequality (5) that in the domain  $D_0$ ,  $a_\varepsilon p^2 \leq 4M_7$ . Obviously, this is sufficient for the proof of (4), since  $x_0$  is an arbitrary point.

**Lemma 3.** In the domain  $M_3 \leq t \leq T$ ,

$$|u_{x_i}| \leq M_9. \quad (8)$$

The proof of this lemma is carried out in exactly the same way as that of Lemma 2, taking into account that for  $0 < M_3 \leq t \leq T$  Nash's inequality <sup>(2)</sup> holds:

$$|u(t_1, x_1) - u(t_2, x_2)| \leq M_{10}|x_1 - x_2|^\beta + M_{11}|t_1 - t_2|^{\beta/4} \quad (9)$$

$0 < \beta < 1$ ,  $\beta$  depending only on  $M_1$ , which is also valid for nonhomogeneous equations. By virtue of (9), every point  $(t_0, x_0)$  of  $Q$  can be covered by a parallelepiped  $|x - x_0| \leq M_{12}$ ,  $|t - t_0| \leq M_{13}$ , where the substitution (6), and hence the estimate  $|u_{x_i}|$ , is possible.

**Lemma 4.** The derivatives  $u_{x_i}$ ,  $u_{x_i x_j}$ ,  $u_t$  satisfy in  $Q$  a Hölder condition with Hölder constants and exponent depending only on  $M_1$ .

We obtain the estimate  $|u_{x_i x_j}| \leq M_{14}$  by using the function  $a_\varepsilon(x)$  and the transformation  $u_{x_k} = \varphi(q_k)$ , analogously to the way the estimates of the second derivatives were obtained in the paper <sup>(4)</sup> (pp. 508-511). The estimate  $|u_t| \leq M_{15}$  follows from (1). The functions  $\bar{u} = a_1(x - x_0)[u - u_0]$  and  $\bar{p}^k = a_1(x - x_0)[u_{x_k} - \partial u_0 / \partial x_k]$  in the parallelepiped  $\{|x - x_0| \leq 4, 0 \leq t \leq T\}$  satisfy zero initial and boundary conditions and the equations

$$\sum_{i,j=1}^n a_{ij} \bar{u}_{x_i x_j} = f_1, \quad \sum_{i,j=1}^n a_{ij} \bar{p}_{x_i x_j}^k = f_2, \quad \text{where } |f_1| \leq M_{16}, |f_2| \leq M_{17}.$$

Using these equations and Theorem 1 from the paper <sup>(5)</sup>, we obtain that  $u_{x_i}$  and  $u_{x_i x_j}$  satisfy a Hölder condition. We shall also assume that  $u_0(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  and  $a(t, x, 0, 0) \equiv 0$ .

**Theorem 1.** Under the assumptions indicated above on  $a_{ij}, a, u_0(x)$ , there exists in  $Q$  a unique solution  $u(t, x)$  of equation (1) with condition (2); moreover  $u, u_{x_i}, u_t, u_{x_i x_j}$  are bounded in  $Q$  and satisfy a Hölder condition.

The proof of Theorem 1 follows from the Leray-Schauder theorem <sup>(1)</sup> applied to the family of equations

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(t, x, ku) \frac{\partial u}{\partial x_j} \right) + \tilde{a}(t, x, ku, ku_x) u + a(t, x, 0, ku_x) = 0,$$

where

$$0 \leq k \leq 1, \quad \tilde{a}(t, x, u, u_x)u + a(t, x, 0, u_x) \equiv a,$$

with condition (2), if one uses the a priori estimates obtained, which are independent of  $k$ , the results of [6], and the condition on  $u_0(x)$ .

**2. Boundary-value problem.** Consider equation (1) in the cylinder  $D\{\Omega \times [0, T]\}$ , where  $\Omega$  is a certain domain in the space  $x$  with smooth boundary  $\Gamma$ . Denote  $\Gamma \times [0, T]$  by  $S$ . Let  $u_0(t, x)$  be a sufficiently smooth function in  $D$ . We shall seek a solution  $u(t, x)$  of equation (1) with the condition

$$u|_{\Omega+S} = u_0(t, x), \quad \text{where } |u_0| \leq K_0. \quad (10)$$

We assume that for  $u_0(t, x)$  the compatibility condition is fulfilled, i.e.  $u_0$  satisfies equation (1) at the points of  $\Gamma$ . We shall establish a number of a priori estimates for  $u(t, x)$ . By  $K_l$  ( $l = 1, 2, \dots$ ) we shall denote positive constants depending on  $c, T$ , on the maxima of the moduli of  $u_0(t, x)$  and its derivatives,  $a_{ij}, a$  and their derivatives in  $D$  for  $|u| \leq K_0 + A/|c| = K_1$ , and also on the smoothness of  $\Gamma$ .

**Lemma 5.** For the solution  $u(t, x)$  of problem (1), (10) in  $D$ , the estimates

$$|u(t, x)| \leq K_1, \quad |u(t, x) - u_0(t, x)| \leq K_2 t \quad (11)$$

hold. On the boundary  $S$  of the cylinder  $D$ ,  $|u_{x_i}| \leq K_3$ . At all points of  $D$ ,

$$|u(t, x) - u_0(t, x)| \leq K_4 \rho(t, x), \quad (12)$$

where  $\rho(t, x)$  is the distance of the point  $(t, x)$  to  $S$ .

The estimates (11) follow from the maximum principle. The estimate of  $|u_{x_i}|$  on  $S$  and (12) are obtained by a method close to that used in § 5 of [3].

**Lemma 6.** Let  $\rho_1(t, x)$  be the distance of  $(t, x)$  to  $S + \Omega$ . In the domain  $\rho_1(t, x) \leq K_5$ ,

$$|u_{x_i}| \leq K_6, \quad |u_{x_i x_j}| \leq K_7, \quad |u_t| \leq K_8. \quad (13)$$

The estimate (13) for  $|u_{x_i}|$  is proved in exactly the same way as Lemma 2, taking into account (12) and (11). The estimate of  $|u_{x_i x_j}|$  on  $S$  may be obtained analogously to the way the lemma in [4] was proved. Using this result, the estimate for  $|u_{x_i x_j}|$  for  $\rho_1 \leq K_5$  is obtained in the same way as in Lemma 4.

We extend the function  $u(t, x)$  for all  $x$  and  $0 \leq t \leq T$  so that  $u, u_{x_i}, u_t, u_{x_i x_j}$  do not exceed in absolute value a certain constant  $K_9$ . The obtained function  $u(t, x)$  satisfies in  $Q$  the equation

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(t, x, u) \frac{\partial u}{\partial x_j} \right) - \frac{\partial u}{\partial t} = f_3,$$

where  $|f_3| \leq K_{10}$ . Using the theorem of J. Nash [2], we obtain that in the cylinder  $D$

$$|u(t_1, x_1) - u(t_2, x_2)| \leq K_{12}|x_1 - x_2|^\gamma + K_{11}|t_1 - t_2|^{\gamma/4}. \quad (14)$$

Using (14) and the results of [5], we establish, as for the solution of the Cauchy problem, that  $u_{x_i}, u_{x_i x_j}, u_t$  are bounded in  $D$  and satisfy a Hölder condition with exponent and Hölder constants depending only on  $K_i$ . Here we assume that  $u_0 \in C^3$  in  $D$ ,  $a_{ij} \in C^{3+\alpha}$ ,  $a \in C^{2+\alpha}$  for  $|u| \leq K_1$ ,  $0 < \alpha \leq 1$ , and the boundary  $\Gamma$  is given by functions of local coordinates of class  $C^3$ .

**Theorem 2.** Under the indicated assumptions on  $a_{ij}, a, u_0(t, x)$ , and  $\Gamma$ , there exists in  $D$  a unique solution  $u(t, x)$  of the boundary-value problem (1), (10), and  $u_{x_i}, u_t, u_{x_i x_j}$  satisfy a Hölder condition in  $D$ .

The proof of this theorem is carried out in exactly the same way as that of Theorem 1, on the basis of the a priori estimates indicated above for the solution of problem (1), (10).

**3. Approximate solution of the Cauchy problem (1), (2) and the boundary-value problem (1), (10). Generalized solutions of these problems.** Consider problem (1), (2). Let  $u_0(x)$  be bounded and summable with square in  $R\{-\infty < x < +\infty\}$ . We construct the approximate solution  $u_\tau(t, x)$ ,  $0 < \tau \leq 1$ , of problem (1), (2) as the solution of the linear equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(t, x, u(t-\tau, x)) \frac{\partial u}{\partial x_j} \right) + \\ &+ \tilde{a}(t, x, u(t-\tau, x), u_x(t-\tau, x))u + a(t, x, 0, u_x(t-\tau)) = 0 \end{aligned} \quad (15)$$

with the condition

$$u_\tau = u_{0\tau}(x) \quad \text{for } -\tau \leq t \leq 0, \quad (16)$$

where  $u_{0\tau}$  are smooth functions, uniformly bounded in  $R$  with respect to  $\tau$ , and  $u_{0\tau} \rightarrow u_0(x)$  in  $\mathcal{L}_2(R)$  as  $\tau \rightarrow 0$ . The solutions  $u_\tau(t, x)$  are uniformly bounded in  $Q$  by virtue of the maximum principle. The norms of  $u_\tau$  and  $\partial u_\tau / \partial x_i$  in  $\mathcal{L}_2(Q)$ , and of  $u_\tau$  in  $\mathcal{L}_2(R)$  for any  $t$ , are also bounded with respect to  $\tau$ . On the basis of J. Nash's theorem<sup>2</sup>, the family  $u_\tau$  is equicontinuous for  $T \geq t \geq t_0 > 0$ . It is easy to show that the function  $u(t, x) = \lim u_\tau$  as  $\tau \rightarrow 0$  is a generalized

solution of problem (1), (2) in the following sense: 1)  $u(t, x)$  is bounded in  $Q$  and has generalized derivatives  $u_{x_i}$  from  $\mathcal{L}_2(Q)$ ; 2)  $u(t, x)$  satisfies the Hölder condition for  $t \geq t_0 > 0$  for any  $t_0$ ; 3) for every  $t_0$  the integral identity holds

$$\iint_{T \geq t \geq t_0} \left[ \frac{\partial F}{\partial t} u - \sum_{i,j=1}^n a_{ij} u_{x_i} F_{x_j} + aF \right] dx dt + \int_{t=t_0} F(t_0, x) u(t_0, x) dx = 0$$

for every smooth finite function  $F(t, x)$ ; 4)  $u(t, x)$  converges in the mean to  $u_0(x)$  as  $t \rightarrow 0$ .

**Theorem 3.** The generalized solution  $u(t, x)$  of problem (1), (2) exists, is unique, and is the limit of the solutions  $u_\tau$  of the linear equations (15) with condition (16).

The uniqueness of the generalized solution  $u(t, x)$  is proved by a method close to Holmgren's method, using the results of paper <sup>6</sup>.

It is easy to show that if  $u_0(x) \in \mathcal{L}_2(R)$ , then the classical solution of problem (1), (2) constructed in Theorem 1 is also a generalized solution of this problem. Similarly, one can define the generalized solution of the boundary-value problem (1), (10) and prove its existence and uniqueness.

We note that, in exactly the same way as Theorems 1 and 2 were proved, the Cauchy problem and the boundary-value problem can be solved in the case when  $a_{ij} \equiv a_{ij}(t, x, u, u_x)$ , under certain restrictions on the dependence of  $a_{ij}$  on  $u_x$ , as well as boundary-value problems for the corresponding quasilinear equations of elliptic type.

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## CITED LITERATURE

1. J. Leray, J. Schauder, *UMN*, 1, issue 3-4 (13-14), 71 (1946).
2. J. Nash, *Am. J. Math.*, 80, No. 4, 931 (1958); *Matematika Collection*, 4, 1, 31 (1960).
3. O. A. Oleinik, T. D. Venttsel, a) *DAN*, 97, No. 4, 605 (1954); b) *Matem. sborn.*, 41 (83), No. 1, 105 (1957).
4. T. D. Venttsel, *Matem. sborn.*, 41 (83), No. 4, 499 (1957).
5. A. Friedman, *J. Math. and Mech.*, 9, No. 4, 539 (1960).
6. A. Friedman, *J. Math. and Mech.*, 7, No. 5, 771 (1958).

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