ON A NONSTATIONARY PROBLEM OF MAGNETOHYDRODYNAMICS

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Abstract

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ON A NONSTATIONARY PROBLEM OF MAGNETOHYDRODYNAMICS

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In the present work we consider the flow past a flat plate by a nonstationary stream of a viscous incompressible fluid possessing finite electrical conductivity. Perpendicular to the plane of the plate there acts an external constant magnetic field with induction B_0. The induced magnetic field arising in the fluid is assumed to be negligibly small in comparison with the external magnetic field. The equations of magnetohydrodynamics under the boundary-layer assumptions (according to Rossow $(^1)$) will have the form

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0; \tag{1}$$

$$\nu\frac{\partial^2 v_x}{\partial y^2}-\frac{\partial v_x}{\partial t}=v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\sigma B_0^2}{\rho}v_x; \tag{2}$$

$$\alpha\frac{\partial^2 T}{\partial y^2}-\frac{\partial T}{\partial t}=v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}-\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2-\frac{\sigma B_0^2}{\rho c_p}v_x-\frac{1}{\rho c_p}\frac{\partial p}{\partial x}-\frac{1}{\rho c_p}v_x\frac{\partial p}{\partial x}, \tag{3}$$

where v_x, v_y are the components of the velocity vector in the boundary layer; B_0 is the magnitude of the external magnetic-induction vector; p is the pressure; ρ is the density; σ is the coefficient of electrical conductivity; c_p is the heat capacity at constant pressure; α is the coefficient of thermal diffusion; ν is the coefficient of viscosity; T is the temperature.

We shall assume that the Prandtl number $\mathrm{Pr}=\nu/\alpha=1$. If we denote $E=c_{pT}+v_x^2/2$, then from (2) and (3) we obtain

$$\nu\frac{\partial^2 E}{\partial y^2}-\frac{\partial E}{\partial t}=v_x\frac{\partial E}{\partial x}+v_y\frac{\partial E}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial t}. \tag{4}$$

Thus, the problem consists in determining v_x, v_y, E from equations (1), (2), and (3) under the following initial and boundary conditions:

$$v_x(x,y,0)=v_x^0(x,y),\qquad v_y(x,y,0)=v_y^0(x,y),$$
$$v_x(x,0,t)=v_y(x,0,t)=0,\qquad v_x(x,\infty,t)=u_0(x,t),$$
$$E(x,y,0)=E^0(x,y),\qquad E(x,0,t)=E_{\text{pl}}(x,t),\qquad E(x,\infty,t)=E_\infty(x,t). \tag{5}$$

Here it is understood that outside the boundary layer the motion is potential with velocity $v_x = u_0(x,t)$, and the pressure does not depend on y and is determined from the equation of potential motion. We denote

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=f(x,t),\qquad \frac{1}{\rho}\frac{\partial p}{\partial t}=f_1(x,t). \tag{6}$$

In the stationary case, analogous problems were considered by Rossow (^{1}) and Scesa (^{2}). In the present work the solution of the problem is reduced to integral ...

equations, which are solved by successive approximations. This method can also be generalized to solutions of problems of magnetic boundary layers of the first and second kind, considered by V. N. Zhigulev (^{3}). If one introduces the Green' s function for the heat-conduction equations

$$G(y,\eta,t)=-\frac{1}{2\sqrt{\pi\nu t}}\exp\left[-\frac{(y-\eta)^2}{4\nu t}\right]+\int_0^t \frac{\exp\left[-\frac{\eta^2}{4\nu\tau}-\frac{y^2}{4\nu(t-\tau)}\right]y}{4\pi\nu\sqrt{\tau(t-\tau)^3}}\,d\tau, \tag{7}$$

then the solution of our problem can be reduced to the following functional equations (^{4}):

$$v_x(x,y,t)=V_1(x,y,t)+\int_0^t d\tau\int_0^\infty\left(v_x\frac{\partial v_x}{\partial x}-\frac{\partial v_x}{\partial \eta}\int_0^\eta \frac{\partial v_x}{\partial x}\,d\eta+v_x\frac{\sigma B_0^2}{\rho}\right)G(y,\eta,t-\tau)\,d\eta; \tag{8}$$

$$E(x,y,t)=V_2(x,y,t)+\int_0^t d\tau\int_0^\infty\left(v_x\frac{\partial E}{\partial x}-\frac{\partial E}{\partial \eta}\int_0^\eta \frac{\partial v_x}{\partial x}\,d\eta\right)G(y,\eta,t-\tau)\,d\eta, \tag{9}$$

where $V_i(x,y,t)$ satisfy the heat-conduction equation

$$\nu\frac{\partial^2 V_i}{\partial y^2}-\frac{\partial V_i}{\partial t}=F_i(x,y,t),\qquad i=1,2, \tag{10}$$

and the boundary conditions (5). In equations (2) and (3) the values

$$v_y = -\int_0^y \frac{\partial v_x}{\partial x}\,dy \tag{11}$$

have been substituted from the continuity equation (1).

It can be shown that if $a \le x \le b$, then for an arbitrary continuous function $\Phi(x, y, t)$ the formula

$$\Phi(x,y,t) = \lim_{z\to 0} \frac{1}{\sqrt{\pi z}} \int_a^b \Phi(\xi, y, t) e^{-(x-\xi)^2/z}\,d\xi, \tag{12}$$

holds when $a < x < b$.

Consider the following functional equations:

$$u = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(u\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\int_0^\eta \frac{\partial u}{\partial \xi}\,d\eta + \frac{\sigma B_0^2}{\rho}u \right) G(y,\eta,t{-}\tau) e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + V_1(x,y,t); \tag{13}$$

$$h = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(u\frac{\partial h}{\partial \xi} - \frac{\partial h}{\partial \eta}\int_0^\eta \frac{\partial u}{\partial \xi}\,d\eta \right) G(y,\eta,t{-}\tau) e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + V_2(x,y,t). \tag{14}$$

Using (12), we obtain the solution of our problem (8) and (9) from the solutions of (13) and (14) for

$$\lim_{z\to 0} u = v_x, \qquad \lim_{z\to 0} h = E, \qquad \delta = 1.$$

Putting $\partial u/\partial x = v$, $\partial u/\partial y = w$, $\partial h/\partial x = \varphi$, $\partial h/\partial y = \psi$, from (13) and (14) we obtain

$$u = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(uv - w\int_0^\eta v\,d\eta + \frac{\sigma B_0^2}{\rho}u \right) G(y,\eta,t{-}\tau) e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + V_1(x,y,t),$$

$$v = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(uv - w\int_0^\eta v\,d\eta + \frac{\sigma B_0^2}{\rho}u \right) G(y,\eta,t{-}\tau) e^{-(x-\xi)^2/z} \frac{2(\xi - x)}{\sqrt{\pi z^3}}\,d\xi + \frac{\partial V_1}{\partial x},$$

$$w = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(uv - w \int_0^\eta v\, d\eta + \frac{\sigma B_0^2}{\rho} u \right) \frac{\partial G}{\partial y} e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + \frac{\partial V_1}{\partial y},$$

$$h = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(u\varphi - \psi \int_0^\eta v\, d\eta \right) G e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + V_2(x, y, t), \quad (15)$$

$$\varphi = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(u\varphi - \psi \int_0^\eta v\, d\eta \right) G e^{-(x-\xi)^2/z} \frac{2(\xi - x)}{\sqrt{\pi z^3}}\, d\xi + \frac{\partial V_2}{\partial x},$$

$$\psi = \delta \int_0^t d\tau \int_0^\infty d\eta \int_a^b \left(u\varphi - \psi \int_0^\eta v\, d\eta \right) \frac{\partial G}{\partial y} e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} + \frac{\partial V_2}{\partial y}.$$

We seek the solution of this system in the form of series

$$u = \sum_{n=0}^{\infty} \delta^n u_n, \qquad v = \sum_{n=0}^{\infty} \delta^n v_n, \qquad w = \sum_{n=0}^{\infty} \delta^n w_n,$$

$$h = \sum_{n=0}^{\infty} \delta^n h_n, \qquad \varphi = \sum_{n=0}^{\infty} \delta^n \varphi_n, \qquad \psi = \sum_{n=0}^{\infty} \delta^n \psi_n. \quad (16)$$

Substituting (16) into (15), to determine the terms of the series we obtain the following recurrence formulas:

$$u_0 = V_1, \qquad v_0 = \frac{\partial V_1}{\partial x}, \qquad w_0 = \frac{\partial V_1}{\partial y},$$

$$u_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} v_m - w_m \int_0^\eta v_{n-m}\, d\eta + \frac{\sigma B_0^2}{\rho} u_m \right) G e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}},$$

$$v_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} v_m - w_m \int_0^\eta v_{n-m}\, d\eta + \frac{\sigma B_0^2}{\rho} u_m \right) G e^{-(x-\xi)^2/z} \times$$

$$\times \frac{2(\xi - x)}{\sqrt{\pi z^3}}\, d\xi,$$

$$w_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} v_m - w_m \int_0^\eta v_{n-m}\, d\eta + \frac{\sigma B_0^2}{\rho} u_m \right) \frac{\partial G}{\partial y} e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}},$$

$$h_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} \varphi_m - \psi_m \int_0^\eta v_{n-m}\, d\eta \right) G e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}},$$

$$\varphi_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} \varphi_m - \psi_m \int_0^\eta v_{n-m}\, d\eta \right) G e^{-(x-\xi)^2/z} \frac{2(\xi - x)}{\sqrt{\pi z^3}}\, d\xi,$$

$$\psi_{n+1} = \int_0^t d\tau \int_0^\infty d\eta \int_a^b \sum_{m=0}^{n} \left(u_{n-m} \varphi_m - \psi_m \int_0^\eta v_{n-m}\, d\eta \right) \frac{\partial G}{\partial y} e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}}.$$

To prove the convergence of the series, let us note that the following inequalities hold $(4, 5)$:

$$\left| \int_0^t d\tau \int_0^\infty G\, d\eta \right|, \quad \left| \int_0^t d\tau \int_0^\infty \frac{\partial G}{\partial y}\, d\eta \right|, \quad \left| \int_0^t d\tau \int_0^\infty \frac{\partial G}{\partial y}\, y\, d\eta \right| < M\sqrt{t},$$

$$\left| \int_a^b e^{-(x-\xi)^2/z} \frac{d\xi}{\sqrt{\pi z}} \right|, \quad \left| \int_a^b e^{-(x-\xi)^2/z} \frac{2(x-\xi)}{\sqrt{\pi z^3}}\, d\xi \right| < N,$$

$$|V_1|, \quad \left| \frac{\partial V_1}{\partial x} \right|, \quad \left| \frac{\partial V_1}{\partial y} \right|, \quad \left| y \frac{\partial V_1}{\partial y} \right|, \quad \left| \frac{\sigma B_0^2}{\rho} \right| < L,$$

where M, N, and L are constants.

Then

$$|u_n|, \ |v_n|, \ |\omega_n| < (n+2)!\, M^n L^{n+1} N^n t^{n/2} \frac{\left[\Gamma(3/2)\Gamma(1/2)\right]^{n-1}}{\Gamma(n/2+1)},$$

where Γ is the gamma function. Hence we obtain a sufficient criterion for the absolute and uniform convergence of the series for $\delta = 1$.

In a similar way one can prove the convergence of the series $|h_n|$, $|\varphi_n|$, $|\psi_n|$. After finding E and v_x, it is easy to determine the temperature distribution in the boundary layer in the presence of a magnetic field. If there is no magnetic field, then $B_0 = 0$, and we obtain the solution of the problem of ordinary hydrodynamics (4).

From (8) and (9) one can also obtain the solutions which Rossow (1) obtained for the case when $v_x = u(y, t)$, $v_y = v_z = 0$, neglecting the pressure gradient.

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Note: Figure translations are in progress. See original paper for figures.