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MATHEMATICS

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1961

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Abstract

Full Text

MATHEMATICS

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ON THE INDICATOR OF A FUNCTION SUBHARMONIC IN A MULTIDIMENSIONAL SPACE

(Presented by Academician S. N. Bernstein on 25 III 1961)

A function subharmonic in the whole space is called a function of order ρ , if*

$$\overline{\lim}_{R \rightarrow \infty} \ln^+ M_u(R) \cdot (\ln R)^{-1} = \rho < \infty, \quad (1)$$

and of normal type σ at order ρ , if

$$\overline{\lim}_{R \rightarrow \infty} M_u(R) \cdot R^{-\rho} = \sigma < \infty, \quad (2)$$

where $M_u(R)$ denotes the maximum of $u^+(Q)$ in the ball K_R of radius R with center at the origin O .

To characterize the growth of a function $u(Q)$ of finite order and normal type in different directions, the concept of an indicator of growth is introduced, analogously to how this is done for entire functions.

Let \mathbf{x} be a unit vector, and let $u(R\mathbf{x})$ be the value of $u(Q)$ at the end of the vector $R\mathbf{x}$, whose origin is at the point O .

Definition. The **indicator** of a function $u(Q)$, subharmonic in the whole space, of order ρ , is the function

$$h(\mathbf{x}) = \overline{\lim}_{R \rightarrow \infty} u(R\mathbf{x}) \cdot R^{-\rho}. \quad (3)$$

In the present article the properties of such an indicator are investigated and, in particular, a property analogous to the trigonometric convexity of the indicator of an entire function is proved.

Along with the notation already introduced, we shall use the following: E_m is m -dimensional space; $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are vectors of this space, and p, q, r are their absolute values; S_R is the sphere of radius R with center at the point O ; σ_1 is the area of the sphere S_1 .

By F. Riesz' s theorem, to every subharmonic function there corresponds a certain distribution of masses in space. We shall assume (this does not restrict

generality) that some neighborhood of the origin is free of masses. Unless the contrary is specified, by subharmonic functions we shall mean functions subharmonic in the whole space, of finite order, satisfying this condition.

From the definition of the indicator it is clear that the indicator of a subharmonic function of normal type is bounded above. However, in contrast to the indicator of an entire function, the indicator of a subharmonic function is, generally speaking, discontinuous and may even take the value $-\infty$ on a set of vectors everywhere dense in S_1 . One can verify this by taking, as an example, a subharmonic function with masses distributed along the rays $\{\lambda \mathbf{x}_j\}$ ($0 < \lambda < \infty$), where $\{\mathbf{x}_j\}$ is a countable set of vectors everywhere dense in S_1 . The set of points,

$$* f^+(Q) = \max[f(Q), 0].$$

where the indicator of a subharmonic function of normal type is infinite, nevertheless not too large, as the following theorem shows.

Theorem 1. *The indicator of a subharmonic function of normal type is summable over the unit sphere.*

We omit the proof. Let us note that the order of growth, the type, and the indicator can also be defined for a function subharmonic in the cone K_Ω determined by a certain domain Ω on the unit sphere. To this end, in equalities (1), (2), (3) one should regard $u(Q)$ as a subharmonic function in K_Ω , take the maximum $u^+(Q)$ in (1) and (2) over $Q \in K_\Omega \cap S_R$, and in (3) regard $x \in \Omega$.

The indicator $h(\theta)$ of an entire function of finite order (as well as of a function holomorphic inside an angle and, in general, of a subharmonic function of order ρ inside an angle in the plane) has the property of trigonometric convexity, consisting in the following: if $H(\theta)$ is a linear combination of the trigonometric functions $\cos \rho\theta$ and $\sin \rho\theta$, satisfying at the ends of a sufficiently small interval (θ_1, θ_2) the conditions $H(\theta_1) \geq h(\theta_1)$, $H(\theta_2) \geq h(\theta_2)$, then $h(\theta) \leq H(\theta)$ throughout the interval (θ_1, θ_2) (see (1), p. 73). We note that $H(\theta)$ is a solution of the equation $u'' + \rho^2 u = 0$.

The main result of this note is the theorem that the indicator of a subharmonic function of order ρ in a certain cone with vertex at the origin has an analogous property with respect to solutions of the equation

$$L[u] + \rho(\rho + m - 2)u = 0, \tag{4}$$

where m is the dimension of the space, and $L[u]$ is the spherical operator:

$$L[u] = (\sin^2 \theta_2 \cdots \sin^2 \theta_{m-1})^{-1} \frac{\partial^2}{\partial \theta_1^2} u +$$

$$\begin{aligned}
 &+(\sin^2 \theta_3 \cdots \sin^2 \theta_{m-1})^{-1} \sin^{-1} \theta_2 \cdot \frac{\partial}{\partial \theta_2} \sin \theta_2 \frac{\partial}{\partial \theta_2} u + \\
 &\dots\dots\dots \\
 &+ \sin^{2-m} \theta_{m-1} \frac{\partial}{\partial \theta_{m-1}} \sin^{m-2} \theta_{m-1} \frac{\partial}{\partial \theta_{m-1}} u \tag{5}
 \end{aligned}$$

(θ_i are spherical coordinates; $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_k \leq \pi$, $k = 2, \dots, m - 1$).

Theorem 2. *Let $h(x)$ be the indicator of a function $u(Q)$ of finite order ρ and normal type, subharmonic in the cone K_{Ω_1} ; let Ω be a domain on the unit sphere ($\Omega \subset \Omega_1$), bounded by a curve ω having bounded curvature, and suppose that the first boundary-value problem for equation (4) has a unique solution in Ω . Let $\varphi(x)$ be a continuous function defined on ω . If $H_{\Omega}^{\varphi}(x)$ is the solution of equation (4) taking the value $\varphi(x)$ on ω , and if for $x \in \omega$ $\varphi(x) \geq h(x)$, then*

$$H_{\Omega}^{\varphi}(x) \geq h(x)$$

for every $x \in \Omega$.

We shall precede the proof of the theorem with a number of lemmas.

Lemma 1. *Let $G(p, t)$ be the Green function of the Laplace equation for the cone K_{Ω} determined by the domain Ω , and let $Y_1(x)$ be the eigenfunction corresponding to the first eigenvalue λ_1 of the boundary-value problem*

$$L[Y] + \lambda Y = 0, \quad Y(x)|_{x \in \omega} = 0.$$

Let ρ_1 be the positive root of the equation

$$z(z + m - 2) - \lambda_1 = 0.$$

Then, for some $M > 0$, depending only on K_{Ω} , for $p, t \in K_{\Omega}$ and $2p < t$, the inequalities hold:

$$\begin{aligned}
 G(p, t) &\leq MY_1(t^{-1}t) p^{\rho_1} \cdot t^{-\rho_1-m+2}, \\
 \frac{\partial}{\partial n_t} G(p, t) &\leq M \frac{\partial}{\partial n_1} Y_1(t^{-1}t) p^{\rho_1} \cdot t^{-\rho_1-m+1},
 \end{aligned}$$

where $\partial/\partial n_t$ is the normal derivative with respect to t at the point t , belonging to the conical surface K_{ω} ; $\partial/\partial n_1$ is the derivative along the normal to K_{ω} at the point $x = t^{-1} \cdot t (\in S_1)$.

With the aid of Lemma 1, Lemmas 2 and 3 are proved.

Lemma 2. Let $\varphi(t)$ be a continuous function on K_ω satisfying the inequality

$$|\varphi(t)| < At^\rho,$$

where A is some constant. Then the integral

$$I(r) = \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) \varphi(t) d\sigma_t$$

converges and represents a harmonic function with limiting values $\varphi(t)$ on K_ω . Moreover,

$$I(r) \leq Br^\rho,$$

where B is a constant.

Lemma 3. For a subharmonic function in K_Ω ($\Omega \subset \Omega_1$) of order $\rho < \rho_1$, the integral

$$H(r) = \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) u(t) d\sigma_t$$

exists and the inequality holds

$$u(r) \leq \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) u(t) d\sigma_t. \quad (6)$$

The proof of this lemma is based on a generalization of the Phragmén–Lindelöf principle given in the work (2).

We now proceed to the proof of Theorem 2. Write inequality (6) for the subharmonic function $u(Rr)$:

$$u(Rr) \leq \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) u(Rt) d\sigma_t.$$

Dividing by R^ρ and taking $\overline{\lim}$, we obtain

$$r^\rho \overline{\lim}_{R \rightarrow \infty} \frac{u(Rr)}{(Rr)^\rho} \leq \overline{\lim}_{R \rightarrow \infty} \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) t^\rho \frac{u(Rt)}{(Rt)^\rho} d\sigma_t,$$

and, applying Fatou's lemma as $R \rightarrow \infty$ to the function

$$[C - u(Rt)(Rt)^{-\rho}] \frac{\partial}{\partial n} G(r, t) t^\rho,$$

where $C > \sigma$ is the type of $u(Rt)$, we obtain

$$r^\rho h(r^{-1}r) \leq \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) t^\rho h(t^{-1}t) d\sigma_t.$$

Let $\varphi(x)$ be a continuous function satisfying the condition of the theorem. Obviously,

$$\int_{K_\omega} \frac{\partial}{\partial n} G(r, t) t^\rho \cdot h(t^{-1}t) d\sigma_t \leq \int_{K_\omega} \frac{\partial}{\partial n} G(r, t) t^\rho \varphi(t^{-1}t) d\sigma_t = H(r).$$

Since $H(r)$ and $H_\Omega^\varphi(r^{-1}r) r^\rho$ have the same boundary values and are both of order $\rho < \rho_1$, they coincide (see (2)) and, consequently,

$$r^\rho h(x) \leq r^\rho H_\Omega^\varphi(x)$$

for $x \in \Omega$, whence the assertion of the theorem follows.

The property of the indicator expressed by Theorem 2 (which may naturally be called its “subsphericity”) makes it possible to characterize more precisely the analytic properties of the indicator, namely:

Theorem 3. *The indicator of a subharmonic function $u(Q)$ of normal type in E_m is summable over any closed smooth contour ω with bounded curvature that lies in a domain Ω satisfying the requirements of Theorem 2.*

In conclusion I express my deep gratitude to B. Ya. Levin, who posed the questions considered in this paper.

Received 23 III 1961

CITED LITERATURE

1. B. Ya. Levin, *Distribution of zeros of entire functions*, Moscow, 1956.
2. J. Deny, P. Lelong, **224**, 1024 (1947).

Note: Figure translations are in progress. See original paper for figures.

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