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**Abstract**

**Full Text**

**B. LEVSHENKO**

## ON INFINITE-DIMENSIONAL SPACES

*(Presented by Academician P. S. Aleksandrov on 18 II 1961)*

The first part of the note is devoted to a generalization of Hurewicz's theorem on mappings that lower dimension (see <sup>(1)</sup>, p. 127) to weakly infinite-dimensional spaces\*. In the second part of the note there are two examples. The first generalizes an example of a compactum, constructed by Yu. M. Smirnov <sup>(2)</sup>, which is not decomposable into the sum of a countable number of **closed** finite-dimensional sets, but which has transfinite dimension and, consequently, is decomposable into the sum of a countable number of zero-dimensional sets. The second shows that even for compacta the sum theorem is no longer true with only two summands. Along the way it is proved that for the compacta  $Q^\alpha$  of Yu. M. Smirnov <sup>(2)</sup> the small\*\* transfinite dimension assumes arbitrarily large values.

I. We shall call a space  $R$  **weakly infinite-dimensional in the sense of P. S. Aleksandrov** and in this case shall write  $\text{Dim } R < \aleph$ , if for every countable system of pairs of closed sets  $A_i, B_i$  such that  $A_i \cap B_i = \emptyset$  for all  $i$ , there exist closed partitions\*\*\*  $C_i$  between  $A_i$  and  $B_i$  with empty intersection:  $\bigcap C_i = \emptyset$ .

**Theorem 1.** *If a bicompactum  $X$  is the sum of closed sets  $\Phi_\alpha$  of dimension  $\text{Dim } \Phi_\alpha < \aleph$  and such that for every neighborhood  $O\Phi_\alpha$  of any set  $\Phi_\alpha$  there is a smaller neighborhood  $U\Phi_\alpha$  with boundary  $|U\Phi_\alpha|$  of dimension  $\text{Dim } |U\Phi_\alpha| < \aleph$ , then  $\text{Dim } X < \aleph$ .*

**Theorem 2.** *If a bicompactum  $X$  is mapped onto a bicompactum  $Y$  possessing small transfinite dimension, in such a way that  $\text{Dim } f^{-1}y < \aleph$  for every point  $y \in Y$ , then  $\text{Dim } X < \aleph$ .*

**Corollary.** *Every bicompactum possessing small transfinite dimension is weakly infinite-dimensional.*

These basic results admit further generalizations. Preliminarily, let us call a space  $R$  **weakly infinite-dimensional in the sense of Yu. M. Smirnov** and in this case write  $\text{dim } R < \aleph$ , if in the preceding definition we require that there be partitions  $C_i$  with empty **finite** intersection:  $\bigcap_{i < n} C_i = \emptyset$ , where  $n$  depends on the system of pairs  $\{A_i, B_i\}$ .

**Theorem 1'.** *If a strongly paracompact\*\*\*\* space  $X$  is the sum of closed sets  $\Phi_\alpha$  of dimension  $\text{dim } \Phi_\alpha < \aleph$  and such—*

\* Spaces are everywhere assumed to be regular, and mappings continuous.

\*\* Thus, there exist compacta of arbitrarily large transfinite dimension—a fact considered known but not found in the literature. Yu. M. Smirnov proved <sup>(2)</sup> that for the large transfinite dimension introduced by him  $\text{Ind } Q^\alpha = \alpha$ . The equality  $\text{ind } R = \text{Ind } R$  has not yet been proved even for compacta.

\*\*\* A **partition** in a set  $S$  between sets  $A$  and  $B$  is any set  $C$  such that  $S \setminus C = G \cup H$ , where  $G$  and  $H$  are disjoint sets open in  $S$ , with  $A \cap S \subseteq G$  and  $B \cap S \subseteq H$ .

\*\*\*\* A space is **strongly paracompact** if into every open cover of it one can inscribe a star-finite cover <sup>(3)</sup>, i.e. one such that each of its elements intersects only finitely many other elements. Every space with a countable base and even every finally compact (Lindelöf) space is strongly paracompact.

such that for every neighborhood  $O\Phi_\alpha$  of any set  $\Phi_\alpha$  there is a smaller neighborhood  $U\Phi_\alpha$  with boundary  $|U\Phi_\alpha|$  of dimension  $\text{Dim } |U\Phi_\alpha| < \aleph$ , then also  $\text{Dim } X < \aleph$ .

**Theorem 2'.** If  $f$  is a closed continuous mapping of a strongly paracompact space  $X$  onto a space  $Y$  representable as the sum of a countable number of subspaces having transfinite dimension, and if  $\dim f^{-1}(y) < \aleph$  for every point  $y \in Y$ , then  $\text{Dim } X < \aleph$ .

**Corollary.** Every strongly paracompact space having small transfinite dimension is weakly finite-dimensional.

**Proof of Theorem 1'.** Let  $\{A^i, B^i; A_*^i, B_*^i\}$  be a countable system of pairs such that

$$A^i \cap B^i = A_*^i \cap B_*^i = \emptyset.$$

Since  $\dim \Phi_\alpha < \aleph$ , in  $\Phi_\alpha$  there exist partitions  $C_\alpha^i$  between  $A^i$  and  $B^i$  such that

$$\bigcap_{i < n(\alpha)} C_\alpha^i = \emptyset.$$

Since  $X$  is normal,\* there exist neighborhoods  $U\Phi_\alpha$  and partitions  $\bar{C}_\alpha^i$  lying in  $[U\Phi_\alpha]$ , between  $A^i$  and  $B^i$ , such that  $\text{Dim } |U\Phi_\alpha| < \aleph$  and

$$\bigcap_{i < n(\alpha)} \bar{C}_\alpha^i = \emptyset.$$

Into the covering  $\{U\Phi_\alpha\}$  we inscribe a star-finite covering  $\omega$ . It splits into subsystems  $\omega_\lambda$  such that each of them is countable, and the bodies

$$R_\lambda = \tilde{\omega}_\lambda$$

are open-closed and pairwise disjoint <sup>(3)</sup>. Let  $\omega_\lambda = \{U_{\lambda_j}\}$  and let  $a(\lambda_j)$  be such that  $U_{\lambda_j} \subset U\Phi_{a(\lambda_j)}$ . Let

$$V_{\lambda_j} = R_\lambda \cap \Phi_{a(\lambda_j)},$$

and

$$\bar{C}_{\lambda j}^i = R_\lambda \cap \bar{C}_{a(\lambda j)}^i, \quad i \leq n(a) = n(\lambda j).$$

We carry out the proof inside some piece  $R = R_\lambda$ , omitting the index  $\lambda$ . Suppose that in

$$S_m = \bigcup_{j \leq m} |V_j|$$

partitions  $D_m^i$ , where  $i \leq n(m)$ , have been constructed between  $A^i$  and  $B^i$  such that

$$\bigcap_{i \leq n(m)} D_m^i \subset \bigcup_{i \leq m-1} |S_i|$$

and suppose

$$S_m \setminus D_m^i = G_m^i \cup H_m^i,$$

where  $G_m^i$  and  $H_m^i$  are disjoint open neighborhoods in  $S_m$  of the sets  $A^i \cap S_m$  and  $B^i \cap S_m$ . Similarly,

$$|V_{m+1}| \setminus \bar{C}_{m+1}^i = G_{m+1}^i \cup H_{m+1}^i,$$

where

$$G_{m+1}^i \cap H_{m+1}^i = \emptyset, \quad A^i \cap |V_{m+1}| \subset G_{m+1}^i, \quad B^i \cap |V_{m+1}| \subset H_{m+1}^i.$$

The partitions  $D_{m+1}^i$  in  $S_{m+1}$ , for  $i \leq n(m)$ , are obtained as follows:

$$D_{m+1}^i = D_m^i \cup (S_{m+1} \cap \bar{C}_{m+1}^i \setminus S_m) \cup (|S_m| \cap |V_{m+1}| \setminus (G_{m+1}^i \cup H_{m+1}^i)),$$

where

$$G_{m+1}^i = S_{m+1} \setminus (S_m \setminus G_m^i) \setminus (|V_{m+1}| \setminus G_{m+1}^i),$$

$$H_{m+1}^i = S_{m+1} \setminus (S_m \setminus H_m^i) \setminus (|V_{m+1}| \setminus H_{m+1}^i).$$

For  $n(m) < i \leq n(m+1)$ , the partitions  $D_{m+1}^i$  are arbitrary closed partitions in  $S_{m+1}$  between  $A^i$  and  $B^i$  such that

$$D_{m+1}^i \cap |V_{m+1}| = \bar{C}_{m+1}^i.$$

The sum

$$D^i = \bigcup_m D_m^i$$

is a partition in the piece  $R = R_\lambda$  between  $A^i$  and  $B^i$ , and

$$\bigcap_i D^i \subset \bigcup_m |S_m| \subset \bigcup_j |V_j|.$$

Since  $\text{Dim } V_j < \aleph$ , it follows by <sup>(4)</sup> that

$$\text{Dim} \bigcap_i D^i < \aleph,$$

and hence, in  $\bigcap_i D^i$ , there exist partitions  $C_*^i$  between  $A_*^i$  and  $B_*^i$  such that

$$\bigcap_i C_*^i = \emptyset.$$

Then in  $R_\lambda = R$  there exist partitions  $D_*^i$  between  $A_*^i$  and  $B_*^i$  such that

$$D_*^i \cap \bigcap_i D^i = C_*^i.$$

Consequently,

$$\bigcap_i D_*^i \cap \bigcap_i D^i = \emptyset$$

and the pairs  $(A^i, B^i)$ ,  $(A_*^i, B_*^i)$  are separated in the piece  $R_\lambda$  as required. The theorem is proved.

**Proof of Theorem 2'.** Let  $X_i = f^{-1}Y_i$  and  $\text{ind } Y_i \leq \alpha_i$ . We have:

$$X_i = \bigcup_{y \in Y_i} f^{-1}y,$$

where  $\dim f^{-1}y < \aleph$ , and (by the closedness of the mapping) into any neighborhood  $O f^{-1}y$  of each set  $f^{-1}y$  one can insert a neighborhood  $U = U f^{-1}y$ , for which

$$\text{ind } f|U < \alpha_i.$$

By means of induction and Theorem 1' we obtain  $\text{Dim } X_i < \aleph$ . But then (4) also gives  $\text{Dim } X < \aleph$ . The theorem is proved.

\* By virtue of the normality of the regular strongly paracompact space  $X$ , see (3).

Theorems 1 and 2 follow from theorems 1' and 2' because  $\dim R = \text{Dim } R$  for bicompacts (4).

**Corollary.** *The Tikhonov product of a weakly infinite-dimensional bicompactum by a space which is the sum of a countable number of closed sets having small transfinite dimension is weakly infinite-dimensional in the sense of Alexandrov.*

**II. Theorem 3.** *There exist compacta of every countable small transfinite dimension.*

These are the compacta  $Q^\beta$  of Yu. M. Smirnov (2), defined as follows: by  $Q^n$  we denote the  $n$ -dimensional cube; for a limit index  $\beta$ , the compactum  $Q^\beta$  is the Alexandrov compactification of the discrete sum  $\bigcup_{\alpha < \beta} Q^\alpha$ ; for an isolated index  $\beta$  we put

$$Q^\beta = Q^{\beta-1} \times Q^1.$$

The proof is by induction with a complicated inductive hypothesis, using the sum theorem for a finite number of spaces having transfinite dimension.

**Theorem 3'.** *There exist compacta of every countable small transfinite dimension  $\alpha$  which cannot be represented as the sum of a countable number of subcompacta of smaller dimension.*

**Proof.** Let  $I^n$  be the cube of the Hilbert parallelepiped  $I^\omega$ , at whose points all coordinates  $y_i = 0$  for  $i > n$ . Let  $\Phi_1$  be a compactum homeomorphic to  $Q^\alpha$ , lying in  $I^\omega \times I^1$ . Next, by induction we construct compacta  $\Phi_i$  having the following properties: a)  $\Phi_i \subset I^\omega \times I^i$  and  $\text{ind } \Phi_i = \alpha$ ; b)  $\Phi_i \subset \Phi_{i+1}$ ,  $\Phi_i$  is nowhere dense in  $\Phi_{i+1}$ ; c) if

$$(x; y_1, \dots, y_i, 0, 0, \dots) \in \Phi_i,$$

then

$$(x, y_1, \dots, y_i, 0, y_{i+2}, 0, \dots) \notin \Phi_{i+2}$$

for  $y_{i+2} > 0$ , where  $x \in I^\omega$ ,

$$(y_1, y_2, \dots, y_i, 0, \dots) \in I^i,$$

and

$$(y_1, \dots, y_i, 0, y_{i+2}, 0, \dots) \in I^{i+2}.$$

If the compacta  $\Phi_1, \dots, \Phi_n$  have been constructed, then the compactum  $\Phi_{n+1}$  is constructed in the following way. For the numbers

$$\varepsilon_i = 1/3^{i(n+1)}$$

we take finite  $\varepsilon_i$ -nets

$$x_1^i, \dots, x_{N(i)}^i$$

of the compactum  $\Phi_n$ , such that  $x_j^i \notin \Phi_n$ ,

$$\rho(x_j^i, \Phi_{n-1}) > \varepsilon_i/2$$

and

$$\rho(x_{j_1}^i, x_{j_2}^i) > \varepsilon_i.$$

In the hyperplane  $I^\omega \times I^n \times \varepsilon_i$ , lying in  $I^\omega \times I^{n+1}$ , consider the balls  $U_j^i$  of radius  $\varepsilon_i/3$  with centers at the points  $x_j^i$  ( $j \leq N(i)$ ,  $i = 1, 2, \dots$ ), and in the ball  $U_j^i$  take a compactum  $\Phi_{n+1}^{ij}$ , homeomorphic to  $Q^\alpha$  and such that

$$\Phi_{n+1}^{ij} \subseteq (I^\omega \times I^n \times \varepsilon_i) \setminus (I^\omega \times I^{n-1} \times O \times \varepsilon_i).$$

By construction,

$$\Phi_{n+1} = \overline{\bigcup_{i,j} \Phi_{n+1}^{ij}}.$$

Let

$$E = \bigcup \Phi_n.$$

1)  $\text{ind } E = \alpha$ ; 2)  $E$  is semicompact\*; 3) every open subset of  $E$  has dimension  $\alpha$ . By a theorem of E. G. Sklyarenko<sup>(5)</sup>, in view of 2), there exists a compactum  $F^\alpha$ , which is an extension of the set  $E$ , such that

$$\text{ind}(F^\alpha \setminus E) = 0.$$

Slightly modifying E. G. Sklyarenko's construction, in view of the precautions taken by us, one can arrange that

$$\text{ind } F^\alpha = \alpha.$$

Since no compactum  $H$  of dimension  $< \alpha$  from  $F^\alpha$  is anywhere dense in  $F^\alpha$  (in view of 3)),  $F^\alpha$  is not the sum of a countable number of subcompacta of dimension  $< \alpha$ .

**Corollary.** *There exist spaces with a countable base which are the sum of a countable number of zero-dimensional sets, but are not the sum of a countable number of closed sets having transfinite dimension.*

An example of such a set is Nagata's universal space<sup>(6)</sup>.

**Example.** Let us construct a compactum which is the sum of two subcompacta of smaller transfinite dimension than itself. This is the compactum  $Q^{\omega+1}$ . By the preceding and by theorem 5 from<sup>(2)</sup> we have:

$$\text{ind } Q^{\omega+1} = \text{Ind } Q^{\omega+1} =$$

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\* There is a base of open sets with compact boundaries.

$$= \omega_0 + 1.$$

Let\*

$$I_n = \bigcup_{i=0}^{2^n-1} \left( \frac{2i}{2^{n+1}}, \frac{2i+1}{2^{n+1}} \right), \quad J_n = \bigcup_{i=0}^{2^n-1} \left( \frac{2i+1}{2^{n+1}}, \frac{2i+2}{2^{n+1}} \right),$$

$$A = \left[ \bigcup_n (Q^n \times I_n) \right] \quad \text{and} \quad B = \left[ \bigcup_n (Q^n \times J_n) \right].$$

It is easy to see that  $\text{ind } A \geq \omega_0$ ;  $A$  and  $B$  are homeomorphic;  $A \cup B = Q^{\omega+1}$ . We shall show that  $\text{Ind } A \leq \omega_0$ . Let  $\Phi$  be closed in  $A$ , and let  $U$  be a neighborhood, and  $\Phi' = \Phi \cap (\xi \times Q^1)$ . There exists in  $Q^{\omega+1}$  a neighborhood  $V \times G$  of the set  $\Phi'$  such that  $V \subseteq Q^{\omega_0}$  and  $|V| = \emptyset$ , while  $G$  is the sum of a finite number of intervals  $(a_i, b_i)$  such that

$$Q^{\omega_0} \times a_i \subseteq B \setminus A \cup (\xi \times Q^1) = \dot{B}$$

and  $Q^{\omega_0} \times b_i \subseteq \dot{B}$ , and, moreover,

$$[A \cap (V \times G)] \subseteq U.$$

Since

$$\Phi \setminus (V \times G) \subseteq \bigcup_N Q^i,$$

there exists a neighborhood  $W \subseteq U$  of the set  $\Phi \setminus (V \times G)$  with finite-dimensional boundary. Then

$$\Phi \subseteq W \cup (V \times G) \subseteq [W \cup (V \times G)] \subseteq U,$$

and  $W \cup (V \times G)$  has finite-dimensional boundary, as was required to be proved.

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\* ( ) denotes an interval on the line; [ ] denotes closure.

*Note: Figure translations are in progress. See original paper for figures.*

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