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CYBERNETICS AND THE THEORY OF REGULATION

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Abstract

Full Text

CYBERNETICS AND THE THEORY OF REGULATION

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CODING BY MEANS OF FINITE AUTOMATA

(Presented by Academician A. I. Berg on 24 XI 1960)

1. The method of coding usually studied (see ⁽¹⁾ and the bibliography cited there, as well as ⁽²⁾) assigns to each letter of a message (written as a word over the letters of a finite alphabet) some fixed code (a word over the letters of another alphabet), so that the coding can be realized in devices without memory. (In our terminology, the operators of the usual method of coding have weight 1; see below.) On the other hand, in works on the theory of finite automata ^(3,4) operators are studied whose values on individual input letters change from cycle to cycle.

However, the requirement of synchronizing input and output, adopted in these works, is unnatural in the study of coding problems. In this note the question of the mutual uniqueness of coding by means of discrete devices with memory, but without synchronization, is considered. The coded sets of messages can be described by means of sources with a finite number of states ⁽⁵⁾. The criterion of mutual uniqueness is a generalization of the analogous criterion in ⁽²⁾.

2. Let \tilde{A} , \tilde{D} denote the sets of all words over the alphabets $A = \{a_1, a_2, \dots, a_m\}$ and $D = \{d_1, d_2, \dots, d_{m_1}\}$, respectively. An operator P mapping \tilde{A} into \tilde{D} will be called **finitely determined** (abbreviated f.d.) if the following conditions are satisfied: 1) $P(\Lambda) = \Lambda$ (Λ denotes the empty word); 2) for every $\alpha \in \tilde{A}$ there exists an operator P_α such that $P(\alpha\beta) = P(\alpha)P_\alpha(\beta)$ for any $\beta \in \tilde{A}$; 3) the set of distinct P_α , $\alpha \in \tilde{A}$, is finite. The number h of elements of the latter set will be called the **weight** of the operator (cf. ⁽⁴⁾).

Each operator P_α is also f.d. with weight $\leq h$. Denote by P^i , $i = 1, 2, \dots, h$, the distinct operators P_α , where it is assumed that $P^1 = P_\Lambda = P$. Then one can define a function $i' = f(i, j)$ and words $\delta_j^i \in \tilde{D}$ such that

$$P_{a_j}^i = P^{i'}, \quad P^i(a_j) = \delta_j^i, \quad i = 1, 2, \dots, h; \quad j = 1, 2, \dots, m; \quad 1 \leq i' \leq m,$$

and for any nonempty word $\alpha = a_{j_1} a_{j_2} \dots a_{j_l}$ the relation holds

$$P^i(\alpha) = \delta_{j_1}^i \delta_{j_2}^{i'} \dots \delta_{j_l}^{i''}, \tag{1}$$

where $i_1 = i$, $i_{\nu+1} = f(i_\nu, j_\nu)$, $\nu = 1, 2, \dots, l-1$. Conversely, if a function f and words $\delta_j^i \in \tilde{D}$ are given, then relation (1) (together with the requirement $P^i(\Lambda) = \Lambda$) determines a system of f.d. operators P^i .

The product (superposition) of f.d. operators is also an f.d. operator. The connection between f.d. operators and operators realizable in synchronous finite automata is given by the fact that any f.d. operator P can be represented in the form $P = TSR$, where R is an operator of weight 1 mapping \tilde{A} into \tilde{A} , such that $R(a_j) = a_{ja}j \dots a_j$ (n times); S is a boundedly determined open-

operator in the sense of (4), mapping \tilde{A} into \tilde{D}' (where $D' = D \cup \{d_0\}$); T is a weight-1 operator such that $T(d_j) = d_j$ for $j \neq 0$ and $T(d_0) = \Lambda$ (T maps \tilde{D}' into \tilde{D}).

3. A set $\theta \subset \tilde{A}$ will be called **finitely enumerable** (abbreviated f.e.) if there exists a finite alphabet B and a f.d. operator Q , mapping \tilde{B} into \tilde{A} , such that $Q(\tilde{B}) = \theta$. Finite sets of words, including Λ , and also \tilde{A} are finitely enumerable. If one somewhat narrows the definition of a source in (5), then the sets generated by such sources are finitely enumerable. Let $\Phi \subset \tilde{A}$ be such that $\Lambda \in \Phi$. Then any $\alpha \in \tilde{A}$ can be represented in the form $\alpha = \varphi\beta$, where $\varphi \in \Phi$. Among all these representations choose one, $\alpha = \varphi_\alpha\beta_\alpha$, in which the length of the word φ_α is maximal. This representation is unique; denote by U^Φ the operator mapping \tilde{A} into \tilde{A} , defined by the equality $U^\Phi(\alpha) = \varphi_\alpha$.

Lemma. In order that Φ be f.e., it is necessary and sufficient that U^Φ be f.d.

Let P be a f.d. operator mapping \tilde{A} into \tilde{D} . Denote by Γ_P the set of words from \tilde{A} such that $\alpha \in \Gamma_P$ if and only if either $\alpha = \Lambda$, or $P_{\alpha'}(a_j) \neq \Lambda$, where $\alpha = \alpha'a_j$, $a_j \in A$. Let $\theta \subset \tilde{A}$ be such that $\Lambda \in \theta$. Then the operators P and PU^θ coincide on θ , and if P is one-to-one on θ , then $\Gamma_{PU^\theta} = \theta$. Hence it follows:

Theorem 1. In order that the f.d. operator P be one-to-one on θ , it is necessary and sufficient that $\Gamma_{PU^\theta} = \theta$ and the operator PU^θ be one-to-one on Γ_{PU^θ} .

From the lemma and Theorem 1 it follows that the study of the coding of a f.e. set of words by a f.d. operator in the case of mutual uniqueness reduces to the study of the coding, by some f.d. operator P , of the set Γ_P . The latter, however, is also of independent interest. Namely, if P is chosen so that Γ_P is the set of completed messages that interests us, then for $\alpha \in \Gamma_P$ the word $P(\alpha)$ encodes a completed message; if now new letters are successively appended to α (it is assumed that the coding is performed by some device in time), then as long as the continuation of α is not a completed message, its image does not change. But as soon as $\alpha\beta \in \Gamma_P$ begins to hold, the coding device notifies us of the receipt of a new completed message $\alpha\beta$ by the fact that $P(\alpha\beta) \neq P(\alpha)$ begins to hold.

4. We shall call an operator P an N -operator if P is f.d. and for every $\alpha \in \widetilde{A}$ the following holds: either $P_\alpha(\beta) = \Lambda$ for all $\beta \in \widetilde{A}$, or for all nonempty words $\gamma \in \widetilde{A}$, from the equality $P_\alpha(\gamma) = \Lambda$ it follows that $P_{\alpha\gamma} \neq P_\alpha$.

Theorem 2. a) In order that the f.d. operator P be one-to-one on Γ_P , it is necessary that it be an N -operator; b) if P is an N -operator, then Γ_P is f.e.

It follows from the theorem that if a f.d. operator P is one-to-one on Γ_P , then Γ_P is f.e.

Furthermore, according to the theorem, in studying the conditions for mutual uniqueness of P on Γ_P , one may restrict oneself to N -operators.

5. Let the N -operator $P = P^1$ be given by a function f and by the set $\delta_j^i \in \widetilde{D}$ as indicated in item 2 (therefore the operators P_i , $i = 2, \dots, h$, are also given). By a (k, i) -**decoding** of a word $\delta \in \widetilde{D}$ we shall mean a (nonempty) sequence of pairs of indices

$$\begin{pmatrix} i_1 i_2 \dots i_l \\ j_1 j_2 \dots j_l \end{pmatrix}$$

such that $i_1 = k$, $i_{\nu+1} = f(i_\nu, j_\nu)$, $i_l = i$, $j_l = j$, $1 \leq i_\nu \leq h$, $1 \leq j_\nu \leq m$, $\nu = 1, 2, \dots, l$; $\delta = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \dots \delta_{j_l}^{i_l}$, $\delta_j^i =$

$= \delta_{j_1}^{i_1} \neq \Lambda$. Obviously, if we denote $\alpha = a_{j_1} a_{j_2} \dots a_{j_l}$, then $P^k(\alpha) = \delta$, $P_\alpha^k = P^{i'}$, where $i' = f(i, j)$, and if $i = 1$, then $\alpha \in \Gamma_P$. We shall call a k -decoding of the form $\begin{pmatrix} i \\ j \end{pmatrix}$ a $\begin{pmatrix} k, i \\ j \end{pmatrix}$ -decoding for certain i, j , $1 \leq i \leq h$; $1 \leq j \leq m$. Any word $\delta \in \widetilde{D}$ can have only a finite number of decodings, for the finding of which it is not difficult to give an effective procedure. The operator $P = P^1$ is one-to-one on Γ_P if and only if, for every $i = 1, 2, \dots, h$, any word $\delta \in \widetilde{D}$ has no more than one i -decoding.

Similarly to (2), in order to formulate a necessary and sufficient criterion for one-to-one correspondence, we construct a certain directed graph G , whose “admissible” paths will correspond to words having at least two i -decodings. Denote: $\mathfrak{A} = \{1, 2, \dots, h\}$; \mathfrak{M} is the set of quadruples of the form $\begin{pmatrix} k, \varepsilon, i \\ j \end{pmatrix}$, where $1 \leq k \leq h$, and the word ε is a nonempty proper suffix of the word δ_j^i . As the set of vertices of the graph G we take $\mathfrak{A} \cup \mathfrak{M} \cup \{Z\}$, where a “terminal” vertex Z has been introduced.

The vertices of the graph are joined by (directed) edges according to the following rules: 1) no edge has elements of \mathfrak{A} as its end-points, and no edge starts at Z ; 2) a vertex $i \in \mathfrak{A}$ is joined to $\begin{pmatrix} k, \varepsilon, i' \\ j' \end{pmatrix} \in \mathfrak{M}$ if and only if $\delta_{j'}^{i'}$ has an $\begin{pmatrix} i, i' \\ j' \end{pmatrix}$ -decoding, and the word η such that $\delta_{j'}^{i'} = \eta\varepsilon$ ($\eta \neq \Lambda$) has an $\begin{pmatrix} i, i'' \\ j'' \end{pmatrix}$ -decoding

such that $k = f(i'', j'')$; 3) a vertex $i \in \mathfrak{A}$ is joined to Z if and only if there exist pairs of indices $\binom{i'}{j'}$ and $\binom{i''}{j''}$ (not necessarily distinct) such that $\delta_{j'}^{i'} = \delta_{j''}^{i''}$, for which there are two distinct $\binom{i, i'}{j'}$ - and $\binom{i, i''}{j''}$ -decodings of the word $\delta_{j'}^{i'}$; 4) a vertex $\binom{k, \varepsilon, i}{j} \in \mathfrak{M}$ is joined to $\binom{k', \varepsilon', i'}{j'} \in \mathfrak{M}$ if and only if $k' = f(i, j)$ and there is a $\binom{k, i'}{j'}$ -decoding of the word $\varepsilon\varepsilon'$; 5) $\binom{k, \varepsilon, i}{j} \in \mathfrak{M}$ is joined to Z if and only if there exists a k -decoding of the word ε .

All these connections are effectively found by a finite number of decodings of a finite number of words.

Theorem 3. *In order that the N -operator P be one-to-one on Γ_P , it is necessary and sufficient that the graph G contain no paths joining one of the vertices of \mathfrak{A} with the vertex Z .*

Introduce the notation: H is the maximum of the lengths of the $\binom{k, i}{j}$ -decodings of the word δ_j^i over all possible i, j, k ; M is the maximum of the lengths of the words δ_j^i over i, j ; N is the number of elements in \mathfrak{M} . From the existence of a path in the graph there follows the existence of a path without self-intersections joining the same vertices. Similarly to (2), to each admissible (i.e., leading from \mathfrak{A} to Z) path one can assign a pair of preimages from Γ_P having one and the same image in \widetilde{D} . Counting the lengths of these words leads to the following theorem.

Theorem 4. *The N -operator P is one-to-one on Γ_P if P is one-to-one on the set of words from Γ_P whose lengths do not exceed*

$$h - 1 + 2([N/2](M - 1) + 1) \cdot H$$

(where $[x]$ denotes the integer part of the number x).

This estimate can be sharpened at the cost of complicating it. The case of one-to-one coding of the whole set \widehat{A} is a special case admitting certain simplifications.

Let us note that the constructed graph also makes it possible to judge the presence of (infinite) messages with "infinite delay" (1) (depending on whether it contains cycles).

As possible applications of the described methods of coding with memory to statistical information theory, one may point to the case in which the source of messages is nonstationary.

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Note: Figure translations are in progress. See original paper for figures.

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