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Abstract

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ON THE APPROXIMATE SOLUTION OF VOLTERRA EQUATIONS OF THE SECOND KIND BY THE METHOD OF ITERATIONS

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The present work is devoted to questions of the approximate solution of linear Volterra integral equations of the second kind. The difference between the result obtained here and those of (^{1}, ^{2}) consists in the fact that the order of the estimate of the difference between the true and approximate solutions improves as the smoothness of the kernel and of the free term increases. This is achieved by applying V. S. Ryaben' kii' s interpolation formulas (^{3}), constructed with the aid of number-theoretic grids.

In (^{3}) interpolation formulas are derived for functions $\Phi(x_1, \dots, x_s)$ belonging to the class $E_s^\alpha(C)$, i.e., to the class of functions defined on the unit s-dimensional cube

$$\Phi(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} C(m_1, \dots, m_s) \exp[2\pi i (m_1 x_1 + \dots + m_{sx} s)],$$

for whose Fourier coefficients the estimate

$$|C(m_1, \dots, m_s)| \le \frac{C}{(\bar{m}_1 \cdots \bar{m}_s)^\alpha},$$

is valid, where $\alpha > 1$; $\bar{m} = 1$ when $m = 0$ and $\bar{m} = |m|$ when $m \ne 0$.

We shall say that, on a closed s-dimensional domain D, a function $F(x_1, \dots, x_s)$ belongs to the class $H_s^\alpha(C)$ if, for $(x_1, \dots, x_s) \in D$, the derivatives

$$\partial^k F(x_1, \dots, x_s) / \partial x_1^{\gamma_1} \cdots \partial x_s^{\gamma_s},$$

where $0 \le k \le \alpha s,\ 0 \le \gamma_i \le \alpha,\ \gamma_1 + \dots + \gamma_s = k$, are continuous and bounded in absolute value by the constant C.* We shall slightly modify V. S. Ryaben' kii' s interpolation formulas as applied to our purposes.

Let $F \in H_s^\alpha$ on the unit s-dimensional cube. On the cube $-\frac{1}{2} \leq x_1, \dots, x_s \leq \frac{3}{2}$ introduce a function $F^*(x_1, \dots, x_s)$ such that $F^* \in H_s^\alpha$ on the domain $-\frac{1}{2} \leq x_1, \dots, x_s \leq \frac{3}{2}$ and $F^*(x_1, \dots, x_s) \equiv F(x_1, \dots, x_s)$ for $0 \leq x_1, \dots, x_s \leq 1$.

On the interval $[0, 1]$ introduce a function $\tau(x)$ satisfying the following conditions:

1) $d^k\tau(0)/dx^k = d^k\tau(1)/dx^k = 0,\ k = 0, 1, 2, \dots, \alpha - 2$;

2) $\tau(x) \equiv 1$ for $\frac{1}{4} \leq x \leq \frac{3}{4}$;

3) the derivative $d^\alpha\tau(x)/dx^\alpha$ is discontinuous.

In what follows we shall assume $\tau(x)$ to be periodically extended to the whole axis.

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* In those cases where there is no need to specify the constant C, we shall write simply E_s^α and H_s^α.

Let $N > s$ be prime, and let $a_1, \dots, a_s$ be the optimal coefficients of (4), and

$$\psi_k(x_1, \dots, x_s) = \sum_{\overline{m}1 \cdots \overline{m}_s < \sqrt{N}} \exp\left[2\pi i \left(m_1\left(x_1 - \frac{ka_1}{N}\right) + \cdots + m_s\left(x_s - \frac{ka_s}{N}\right)\right)\right].$$

Lemma. If, on the unit s-dimensional cube, $F \in H_s^\alpha(C)$ and $\alpha > 1$, then for any $\varepsilon > 0$ the estimate

$$\left| F(x_1, \dots, x_s) - \frac{1}{N}\sum_{k=1}^{N} \tau\left(\frac{ka_1}{N}\right) \cdots \tau\left(\frac{ka_s}{N}\right) \psi_k\left(\frac{x_1}{2} + \frac{1}{4}, \dots, \frac{x_s}{2} + \frac{1}{4}\right) \right.$$
$$\left. \times F^*\left(2\left\{\frac{ka_1}{N}\right\} - \frac{1}{2}, \dots, 2\left\{\frac{ka_s}{N}\right\} - \frac{1}{2}\right) \right| \leq \frac{CC_1^s}{N^{\frac{\alpha-1}{2} - \varepsilon}},$$

where the constant C_1 depends on α and ε.*

Proof. Let $\Phi \in E_s^\alpha(C_0)$ and $\alpha > 1$; then for any $\varepsilon > 0$ there exists a constant $C_2(\alpha, \varepsilon)$ such that, for prime $N > s$ and for $a_1, \dots, a_s$ the optimal coefficients, the estimate

$$\left| \Phi(x_1, \dots, x_s) - \frac{1}{N}\sum_{k=1}^{N} \psi_k(x_1, \dots, x_s)\Phi\left(\left\{\frac{ka_1}{N}\right\}, \dots, \left\{\frac{ka_s}{N}\right\}\right) \right| \leq \frac{C_0 C_2^s}{N^{\frac{\alpha-1}{2} - \varepsilon}}. \quad (1)$$

The proof of this estimate is obtained by a slight modification of the arguments of paper (3).

As usual, by integration by parts one can show that if, on the unit s-dimensional cube, the function $F^{**}(x_1, \dots, x_s)$ is defined by $F^{**}(x_1, \dots, x_s) = \tau(x_1) \cdots \tau(x_s) F^{*}(2x_1 - \frac{1}{2}, \dots, 2x_s - \frac{1}{2})$, then $F^{**}(\{x_1\}, \dots, \{x_s\}) \in E_s^{\alpha}$. Applying estimate (1) to $F^{**}(\{x_1\}, \dots, \{x_s\})$ for $\frac{1}{4} \le x_1, \dots, x_s \le \frac{3}{4}$, we obtain

$$\Bigg| F^{*}\left(2x_1 - \frac{1}{2}, \dots, 2x_s - \frac{1}{2}\right) -$$

$$\frac{1}{N} \sum_{k=1}^{N} \tau\left(\frac{ka_1}{N}\right) \cdots \tau\left(\frac{ka_s}{N}\right) \psi_k(x_1, \dots, x_s) F^{*}\left(2\left\{\frac{ka_1}{N}\right\} - \frac{1}{2}, \dots, 2\left\{\frac{ka_s}{N}\right\} - \frac{1}{2}\right)\Bigg| \le$$

$$\le \frac{CC_1^s}{N^{\frac{\alpha-1}{2}-\varepsilon}}.$$

Making the substitution $x_l = y_l/2 + 1/4$ for $0 \le y_l \le 1$, $l = 1, \dots, s$, completes the proof of the lemma.

It follows from the preceding argument that the extension of $F(x_1, \dots, x_s)$ precisely to the cube $-\frac{1}{2} \le x_1, \dots, x_s \le \frac{3}{2}$ is not essential. One could extend F to the cube $-\delta \le x_1, \dots, x_s \le 1 + \delta$, where $\delta > 0$ is arbitrary.

We now turn to computing an approximate solution of the Volterra equation of the second kind. Consider the equation

$$\varphi(x) = \int_0^x K(x, y) \varphi(y)\, dy + f(x). \tag{2}$$

* Here $\{z\}$ denotes the fractional part of z.

Suppose that on the unit square $K \in H_2^{\alpha}$ and on the unit interval $f \in H_1^{\alpha}$, where $\alpha > 1$. Denote

$$A_{k,s} = \tau\left(\frac{ka_1}{N}\right) \cdots \tau\left(\frac{ka_s}{N}\right) \sum_{\bar{m}_1 \cdots \bar{m}_s < \sqrt{N}} \exp\left[2\pi i \left(m_1\left(\frac{1}{4} - \frac{ka_1}{N}\right) + \dots + m_s\left(\frac{1}{4} - \frac{ka_s}{N}\right)\right)\right] \times$$

$$\times [\pi i(m_1 + \dots + m_s),\ \pi i(m_1 + \dots + m_{s-1}), \dots, \pi i m_1,\ 0],$$

where $[y_0, \dots, y_s]$ is the divided difference of order s of the function $W(y) = e^{yx}$ at the points $y_0, \dots, y_s$.

Theorem. If $K(x,y)$ and $f(x)$ satisfy the differentiability conditions stated above; $a_1, \dots, a_s$ are optimal coefficients; $N > \alpha^2$ is prime and

$$n = [(\alpha - 1)\ln N / 2\ln\ln N],$$

then for every $\varepsilon > 0$ the equality

$$\varphi(x) - f(x) - \frac{1}{N}\sum_{k=1}^{N}\sum_{s=1}^{n} A_{k,s} K^*\left(x, 2\left\{\frac{ka_1}{N}\right\} - \frac{1}{2}\right) K^*\left(2\left\{\frac{ka_1}{N}\right\} - \frac{1}{2},\ 2\left\{\frac{ka_2}{N}\right\} - \frac{1}{2}\right)\cdots$$

$$\cdots K^*\left(2\left\{\frac{ka_{s-1}}{N}\right\} - \frac{1}{2},\ 2\left\{\frac{ka_s}{N}\right\} - \frac{1}{2}\right) f^*\left(2\left\{\frac{ka_s}{N}\right\} - \frac{1}{2}\right) = O\left(N^{-\frac{\alpha-1}{2}+\varepsilon}\right),$$

holds, where the constant in O depends on α, ε and on the constants bounding the moduli of the derivatives of the kernel and of the free term.

Proof. As is known, under our assumptions concerning the kernel and the free term,

$$\left|\varphi(x) - f(x) - \sum_{s=1}^{n}\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} K(x, x_1)K(x_1, x_2)\cdots K(x_{s-1}, x_s) f(x_s)\, dx_s\right| \le \frac{C_3^n}{n!}, \tag{3}$$

where C_3 does not depend on n.

Using the representation for the divided difference from (5), we obtain

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} \psi_k\left(\frac{x_1}{2} + \frac{1}{4}, \dots, \frac{x_s}{2} + \frac{1}{4}\right) dx_s =$$

$$= \sum_{\bar{m}_1 \cdots \bar{m}_s < \sqrt{N}} \exp\left[2\pi i\left(m_1\left(\frac{1}{4} - \frac{ka_1}{N}\right) + \cdots + m_s\left(\frac{1}{4} - \frac{ka_s}{N}\right)\right)\right] \times$$

$$\times \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} \exp[\pi i(m_1 x_1 + \cdots + m_{sx} s)]\, dx_s =$$

$$= \sum_{\bar{m}_1 \cdots \bar{m}_s < \sqrt{N}} \exp\left[2\pi i\left(m_1\left(\frac{1}{4} - \frac{ka_1}{N}\right) + \cdots + m_s\left(\frac{1}{4} - \frac{ka_s}{N}\right)\right)\right] \times$$

$$\times\, [\pi i(m_1 + \cdots + m_s),\ \pi i(m_1 + \cdots + m_{s-1}), \dots, \pi i m_1,\ 0].$$

Therefore, taking into account the form of the integrand, by virtue of the lemma we have

$$\Bigg|\int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} K(x,x_1)K(x_1,x_2)\cdots K(x_{s-1},x_s)f(x_s)\,dx_s -$$

$$-\frac{1}{N}\sum_{k=1}^{N} A_{k,s} K^*\left(x, 2\left\{\frac{ka_1}{N}\right\}-\frac{1}{2}\right) K^*\left(2\left\{\frac{ka_1}{N}\right\}-\frac{1}{2}, 2\left\{\frac{ka_2}{N}\right\}-\frac{1}{2}\right)\cdots$$

$$\cdots K^*\left(2\left\{\frac{ka_{s-1}}{N}\right\}-\frac{1}{2}, 2\left\{\frac{ka_s}{N}\right\}-\frac{1}{2}\right) f^*\left(2\left\{\frac{ka_s}{N}\right\}-\frac{1}{2}\right) \leq \frac{C_4^s}{s!\,N^{\frac{\alpha-1}{2}-\varepsilon}}, \quad (4)$$

where C_4 depends on α, ε and on the constants bounding the moduli of the derivatives of the kernel and of the free term.

By virtue of inequalities (3), (4) and the choice of n made above, the theorem is proved.

Remark 1. Similar results with the estimate $O\left(N^{-\frac{\alpha-1}{2}+\varepsilon}\right)$ are valid for multidimensional linear Volterra equations of the second kind and for equations in which only part of the integrations is carried out over variable limits.

Remark 2. Arguments analogous to those set forth above are valid also for other interpolation formulas in the class E_s^α. Namely, suppose that for $\Phi(x_1,\ldots,x_s) \in E_s^\alpha(C)$ the formula

$$\left|\Phi(x_1,\ldots,x_s) - \frac{1}{N}\sum_{k=1}^{N}\sum_{\gamma_1=0}^{\alpha-2}\cdots\sum_{\gamma_s=0}^{\alpha-2} \psi'_{k,\gamma_1,\ldots,\gamma_s}(x_1,\ldots,x_s)\,\frac{\partial^{\gamma_1+\cdots+\gamma_s}\Phi\left(\xi_1^{(k)},\ldots,\xi_s^{(k)}\right)}{\partial x_1^{\gamma_1}\cdots\partial x_s^{\gamma_s}}\right| \leq \frac{s!\,CC_5^s}{N^\beta},$$

is valid, where $\psi'_{k,\gamma_1,\ldots,\gamma_s}(x_1,\ldots,x_s)$ are certain known functions; $(\xi_1^{(k)},\ldots,\xi_s^{(k)})$ are the interpolation nodes, and C_5 depends only on α and β. Then, under the notation

$$A'_{k,s} = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} \psi'_{k,\gamma_1,\ldots,\gamma_s}\left(\frac{x_1}{2}+\frac{1}{4},\ldots,\frac{x_s}{2}+\frac{1}{4}\right) dx_s,$$

with the previous assumptions concerning the kernel and the free term of equation (2), and $n = [\beta \ln N/\ln\ln N]$, the equality

$$\varphi(x) - f(x) - \frac{1}{N}\sum_{k=1}^{N}\sum_{\gamma_1=0}^{\alpha-2}\cdots\sum_{\gamma_s=0}^{\alpha-2} A'_{k,s}\,\frac{\partial^{\gamma_1+\cdots+\gamma_s}}{\partial x_1^{\gamma_1}\cdots\partial x_s^{\gamma_s}}\Big\{\tau\Big(\xi_1^{(k)}\Big)\cdots\tau\Big(\xi_s^{(k)}\Big)$$

$$\times K^*\left(x, 2\{\xi_1^{(k)}\} - \frac{1}{2}\right) K^*\left(2\{\xi_1^{(k)}\} - \frac{1}{2}, 2\{\xi_2^{(k)}\} - \frac{1}{2}\right)\cdots$$

$$\cdots K^*\left(2\{\xi_{s-1}^{(k)}\} - \frac{1}{2}, 2\{\xi_s^{(k)}\} - \frac{1}{2}\right) f^*\left(2\{\xi_s^{(k)}\} - \frac{1}{2}\right)\Big\} = O\left(N^{-\beta+\varepsilon}\right)$$

holds.

Of course, such a method for finding the solution of equation (2) makes sense only in the case when, as in the theorem proved above, the computation of the integrals

$$A'_{k,s} = \int_0^{x} dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{s-1}} \psi'_{k,\gamma_1,\ldots,\gamma_s}\left(\frac{x_1}{2} + \frac{1}{4}, \ldots, \frac{x_s}{2} + \frac{1}{4}\right) dx_s$$

does not cause particular difficulties.

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Note: Figure translations are in progress. See original paper for figures.

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