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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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## ON THE DENSITIES OF HYPERIMMUNE SETS

*(Presented by Academician P. S. Novikov on 24 III 1961)*

1. Yu. T. Medvedev <sup>(1)</sup> introduced the following notion of density for subsets of the natural numbers. Let  $E(n)$  be the number of elements of the set  $E \cap [0, n)$ ; in particular,  $E(0) = 0$ .

**Definition 1.** A set  $E_2$  is **greater than or equal in density\*** to a set  $E_1$  if there exists a general recursive function  $\theta$  such that

$$E_2(\theta(n)) \geq E_1(n) \quad (n = 0, 1, 2, \dots).$$

In this case we write  $E_1 \preceq E_2$ .

Through this relation the relations “of the same density” and “less in density” are defined in the natural way. **Density** is defined analogously to cardinality as a class of sets of equal density, i.e. equal in density (or as its abstract characteristic).

Another definition of the indicated notions is given by the following theorem.

**Theorem 1.** *A set  $E_2$  is greater than or equal in density to a set  $E_1$  if and only if there exists a one-to-one mapping of the set  $E_1$  into the set  $E_2$  which is majorized on the set  $E_1$  by some general recursive function, i.e. when there exist functions  $\varphi$  and  $\theta$  such that  $\theta$  is general recursive and*

$$\forall m \forall n (m \in E_1 \ \& \ n \in E_1 \rightarrow \varphi(m) \neq \varphi(n) \ \& \ \varphi(n) \in E_2 \ \& \ \varphi(n) < \theta(n)).$$

It is obvious that the natural numbers have the greatest density, and the empty set the least density. It is also easy to observe that a finite set of  $m$  elements is greater than or equal in density to a finite set of  $n$  elements if and only if  $m \geq n$ . On the other hand, Yu. T. Medvedev proved <sup>(1)</sup> that among infinite sets, hyperimmune sets\*\* and only they have density less than the density of the natural numbers. Therefore, in what follows, we shall consider primarily hyperimmune sets and their densities, which we shall call hyperimmune densities, and only some theorems will concern all subsets of the natural numbers and their densities.

Let us note that one of the characteristic properties of hyperimmune sets (bringing them close to finite sets) is that adding to a hyperimmune set an element not belonging to it increases the density of the set, while deleting at least one element decreases the density <sup>(1)</sup>.

2. **Definition 2.** A density  $\alpha$  is **greater than or equal to** a density  $\beta$  (we write:  $\beta \preceq \alpha$ ) if some (and hence every) set having density  $\alpha$  is greater than or equal in density to some set having density  $\beta$ .

\* In Medvedev's terminology, instead of "greater than or equal in density," one says "not less dense," but the latter is inconvenient because there exist sets incomparable in density.

\*\* For the definition of a hyperimmune set see, for example, <sup>(3)</sup>.

**Theorem 2.** The set of degrees  $H$  (of sets of natural numbers), partially ordered by this relation  $\preceq$ , forms a distributive structure <sup>(2)</sup>.

**Remark 1.** The structure of degrees  $H$  is not a complete structure <sup>(2)</sup>.

**Remark 2.** The degrees of infinite sets form a substructure  $L$  of the structure  $H$ .

**Remark 3.** In the substructure  $L$ , for every hyperimmune degree there exists a degree incomparable with it <sup>(2)</sup>.

3. **Definition 3.** The **successor function** of an infinite set  $E$  is a function  $\psi$ , defined on the set  $E$ , such that

$$\psi(n) = \mu k (k > n \ \& \ k \in E) \quad (n \in E).$$

**Definition 4.** We shall say that a function  $f$  **dominates** a function  $g$  if there exists a natural number  $k$  such that for every natural number  $n > k$  for which  $f(n)$  and  $g(n)$  are defined, one has  $f(n) > g(n)$ .

**Definition 5.** An infinite set is called **rarefied** if its successor function dominates every general recursive function.

**Remark.** Let  $\rho$  be an everywhere-defined function dominating every general recursive function, and let  $n$  be any natural number; then, as is easy to see, the set  $\{n, \rho(n), \rho(\rho(n)), \dots\}$  is a rarefied set. Moreover, one can prove that every infinite set has a rarefied subset.

**Definition 6.** We shall call the sets  $E_1$  and  $E_2$  **strongly different** if their symmetric difference is infinite.

**Theorem 3.** In order that any two strongly different subsets of an infinite set  $E$  have different degrees, it is necessary and sufficient that the set  $E$  be a rarefied set.

**Corollary 1.** Every infinite subset of a rarefied set is a rarefied set.

**Corollary 2.** The set of degrees has the cardinality of the continuum.

**Definition 7.** An infinite set is called **weakly rarefied** if its direct enumeration\* dominates every general recursive function.

It follows from the definition of a weakly rarefied set that every infinite subset of a weakly rarefied set is a weakly rarefied set.

Obviously, a weakly rarefied set is a hyperimmune set. On the other hand, there exist hyperimmune sets that are not weakly rarefied sets. Examples of such sets may be provided by two mutually complementary hyperimmune sets constructed by the author <sup>(2)</sup>.

It is not difficult to prove that every rarefied set is a weakly rarefied set and that there exist weakly rarefied sets that are not rarefied sets. The latter is evident at least from the fact that the union of two weakly rarefied sets is a weakly rarefied set, whereas the union of two rarefied sets may fail to be a rarefied set; for example, the union of an arbitrary rarefied set  $\{n_1, n_2, \dots\}$  with the rarefied set obtained from it  $\{n_1 + 1, n_2 + 1, \dots\}$  is weakly rarefied but not rarefied.

From the following two assertions it follows that the properties of rarefaction and weak rarefaction are, in essence, not properties of sets but properties of degrees, which makes it possible to introduce in a natural way the notions of a **weakly rarefied** and a **rarefied** degree:

\* For the definition of direct enumeration see <sup>(3)</sup>.

- 1) If  $E_1$  is a sparse set, and  $E_2$  is equal to it in density, then  $E_2$  is also a sparse set.
- 2) If  $E_1$  is a weakly sparse set,  $E_2 \preceq E_1$ , and  $E_2$  is an infinite set, then  $E_2$  is a weakly sparse set.

**Remark.** From the fact that  $E_1$  is a sparse set, while  $E_2$  is an infinite set smaller in density than the set  $E_1$ , it does not in general follow that  $E_2$  is a sparse set; for example, if the set  $E_1 = \{n_1, n_2, n_3, n_4, \dots\}$  is sparse and  $n_1 < n_2 < n_3 < n_4 < \dots$ , then the infinite set  $E_2 = \{n_2 - 1, n_2, n_4 - 1, n_4, \dots\}$ , smaller in density than the set  $E_1$ , is not a sparse set (although it is weakly sparse).

**Definition 8.** A set is called **dihyperimmune** if both it itself and its complement are hyperimmune sets.\*

The densities of dihyperimmune sets will be called **dihyperimmune densities**. Let us note that not every set of dihyperimmune density (i.e., one equal in density to some dihyperimmune set) is itself a dihyperimmune set; for example, if the set  $\{n_1, n_2, \dots\}$  is dihyperimmune, then the set  $\{2n_1, 2n_2, \dots\}$  has dihyperimmune density, but is not dihyperimmune.

**Theorem 4.** *The union of a weakly sparse set with a hyperimmune set is a hyperimmune set.*

**Corollary.** For every weakly sparse density there exists a larger hyperimmune density which is not weakly sparse.

It follows from what has been stated that weakly sparse densities are relatively small among other hyperimmune densities, in particular, in comparison with dihyperimmune densities.

**Theorem 5.** \*In the structure  $L$  of densities of infinite sets, the weakly sparse densities form an ideal\*\*.\*

It seems plausible to conjecture that the intersection of all ideals of the structure  $L$  containing all sparse densities coincides with this ideal of weakly sparse densities.

4. In the work of Yu. T. Medvedev <sup>(1)</sup> a problem close to the following was posed: does there exist, between any two comparable hyperimmune densities, an intermediate density?\*\*\* The answer to this question is the following:

**Theorem 6.** *For any such hyperimmune densities  $\alpha$  and  $\beta$ , such that  $\alpha < \beta$ , there exists an intermediate density, i.e., a density  $\gamma$  such that  $\alpha < \gamma < \beta$ .*

Using Theorem 6, it is easy to prove that the densities of all subsets of a hyperimmune set  $E$  do not exhaust all densities smaller than the density of the set  $E$ , i.e., from the fact that  $E_1 \preceq E$ , in general, it does not follow that in  $E$  there is a subset equal in density to the set  $E_1$ .

By the same method as Theorem 6, the following stronger result is proved:

**Theorem 7.** *For any such hyperimmune densities  $\alpha$  and  $\beta$ , such that  $\alpha < \beta$ , there exists a continuum set of pairwise incomparable intermediate densities.*

Of the questions to which the author has not been able to obtain an answer, the following seem interesting:

- 1) Do there exist, besides weakly sparse and dihyperimmune densities, other hyperimmune densities?
- 2) Does there exist a sparse (or at least weakly sparse) set with recursively enumerable complement?

\* Such sets have already been mentioned above.

\*\* For the definition of an ideal of a structure see, for example, <sup>(4)</sup>.

\*\*\* The question posed by Medvedev differs from this in that he dealt with densities of complements of recursively enumerable sets.

- 3) Is the density structure  $H$  a structure with relative pseudocomplements\* (or, as is preferred in mathematical logic, an implicative structure)?

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2. E. A. Pavlova, *Izv. Moldavian Branch of the Academy of Sciences of the USSR*, No. 10 (76) (1960).
3. V. A. Uspenskii, *Lectures on Computable Functions*, Moscow, 1960.
4. G. Birkhoff, *Theory of Structures*, IL, 1952.

\* For the definition of a structure with relative pseudocomplements, see (4).

*Note: Figure translations are in progress. See original paper for figures.*

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