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Abstract

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MATHEMATICS

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THE PROBLEM OF BEST CHEBYSHEV APPROXIMATION IN A COMPLEX DOMAIN WITH ADDITIONAL CONDITIONS ON THE COEFFICIENTS OF THE APPROXIMATING POLYNOMIAL

(Presented by Academician A. N. Kolmogorov on 27 VI 1961)

In the note ⁽⁴⁾ the problem of best Chebyshev approximation of continuous real functions by generalized polynomials whose coefficients are connected by linear relations (equalities and inequalities) was considered. In the present note a complex analogue of this problem is investigated.

1. Let $f(\tau)$, $\varphi_j(\tau)$, $j = 1, 2, \dots, n$, be arbitrary complex functions defined and continuous on the compact set K ;

$$\Phi(\tau) = \{\varphi_1(\tau), \varphi_2(\tau), \dots, \varphi_n(\tau)\}.$$

The problem is posed of finding a generalized polynomial

$$P^*(\tau) = \sum_{j=1}^n d_j^* \varphi_j(\tau) = (\Phi(\tau), d^*),$$

which gives the minimum

$$V(d) = \max_{\tau \in K} \left| f(\tau) - \sum_{j=1}^n d_j \varphi_j(\tau) \right|.$$

under the following additional conditions on the coefficients $d = (d_1, d_2, \dots, d_n)$:

$$(A(t), d) = \sum_{j=1}^n a_j(t) d_j \in G(t), \quad t \in E; \tag{1}$$

$$(A_i, d) = \sum_{j=1}^n a_{ij} d_j = d_i, \quad i = 1, 2, \dots, r. \tag{2}$$

The vector-function $A(t) = \{a_1(t), a_2(t), \dots, a_n(t)\}$ is assumed to be continuous on the compact set E ; the vectors $A_i = (a_{i1}, a_{i2}, \dots, a_{in})$, $i = 1, 2, \dots, r$, are considered linearly independent. By $G(t)$ is meant a family of closed convex plane domains depending on the parameter $t \in E$. The family of domains $G(t)$ is assumed to be continuous in $t \in E$. This means that

$$\lim_{t' \rightarrow t} \rho(\Gamma(t'), \Gamma(t)) = 0, \quad t \in E,$$

where $\Gamma(t)$ is the boundary of $G(t)$, and $\rho(\Gamma(t'), \Gamma(t))$ is the distance between the curves $\Gamma(t)$ and $\Gamma(t')$. For fixed $t \in E$, the constraint (1) is a natural complex analogue of a linear inequality.

We shall call the formulated problem the **Chebyshev problem under the conditions** (constraints) (1), (2). We shall suppose that the constraints (1), (2) of the Chebyshev problem satisfy the following condition.

(R) There exists a vector d , satisfying (2), for which for any $t \in E$ the point $(A(t), d)$ is an interior point of the domain $G(t)$. Hence, in particular, the solvability of the problem follows.

2. Consider an arbitrary point $z \in \Gamma(t) = \tilde{G}(t) - G(t)$. Through this point there pass, generally speaking, two tangents to the domain $G(t)$. We shall call the right (left) one of them that which is situated to the right (left) of the vector $*$, drawn from the point z into $G(t)$. By $n_1(z, t)$ and $n_2(z, t)$ we denote the inward normals to the left and right tangents to $G(t)$ at the point z , respectively. Let $\beta'(z, t) = \arg n_1(z, t)$, $\beta''(z, t) = \arg n_2(z, t)$.

Theorem 1. In order that the polynomial $P^*(\tau) = (\Phi(\tau), d^*)$, satisfying conditions (1), (2), furnish the best approximation to $f(\tau)$ on K under conditions (1), (2), it is necessary and sufficient that there exist points

$$\tau_l \in K, \quad l = 1, 2, \dots, k, \quad k \geq 1,$$

$$|f(\tau_l) - P^*(\tau_l)| = \max_{\tau \in K} |f(\tau) - P^*(\tau)|$$

and

$$t_l \in E, \quad l = 1, 2, \dots, s, \quad (A(t_l), d^*) \in \Gamma(t_l)$$

such that:

$$\text{a) } \sum_{l=1}^k e^{-i\alpha_l} \lambda_l \Phi(\tau_l) + \sum_{l=1}^s e^{-i\beta_l} \mu_l A(t_l) + \sum_{l=1}^r \gamma_l A_l = 0;$$

- b) $\alpha_l = \arg[f(\tau_l) - P^*(\tau_l)]$, $\lambda_l > 0$, $l = 1, 2, \dots, k$; $\beta'(z_l, t_l) \leq \beta_l \leq \beta''(z_l, t_l)$,
 $z_l = (A(t_l), d^*)$, $\mu_l > 0$, $l = 1, 2, \dots, s$;
- c) $1 \leq k + s \leq 2(n - r) + 1$.

Theorem 1 is an analogue of the well-known criterion of P. L. Chebyshev. If the additional constraints (1), (2) are absent, then it becomes the conditions established by E. Ya. Remez ⁽²⁾. In the presence of one constraint—equality (2)—and special assumptions on $f(\tau)$ and $\varphi_j(\tau)$, Theorem 1 was obtained by V. S. Videnskii ⁽³⁾.

The proof of Theorem 1 is essentially based on the result of Theorem 2 of note ⁽⁴⁾. We note that the established criterion makes it possible to construct a numerical method for solving the problem under consideration.

We shall call the collection of points $\tau_i \in K$, $i = 1, 2, \dots, k$; $t_i \in E$, $i = 1, 2, \dots, s$, a **characteristic system of the Chebyshev problem under conditions (1), (2)** if:

- a) every solution of this problem is a solution of the analogous problem in which $K = \{\tau_1, \tau_2, \dots, \tau_k\}$, $E = \{t_1, t_2, \dots, t_s\}$;
- b) no proper subset of the indicated collection of points has property a).

It follows from Theorem 1 that the Chebyshev problem under conditions (1), (2), possessing property (R), has at least one characteristic system. Moreover, for any such system $k \geq 1$, $k + s \leq 2(n - r) + 1$.

3. If the domain $G(t)$ is unbounded, then by passage to the limit one can define the numbers

$$\beta'(t) = \beta'(\infty, t), \quad \beta''(t) = \beta''(\infty, t).$$

Let $\tilde{G}(t)$, $t \in E$, be an arbitrary continuous family of convex closed domains. We shall say that the continuous family of convex closed domains $G(t) \in H_{\tilde{G}}$ if, for every $t \in E$, the domain $G(t)$ is obtained from $\tilde{G}(t)$ by a parallel translation; the constraints (1), (2) of the problem for the given family $G(t)$ satisfy condition (R).

Theorem 2. In order that, for any complex functions $f(t)$, defined and continuous on K , and any families $G(t) \in H_{\tilde{G}}$, the Chebyshev problem under conditions (1), (2) have a unique solution, it is necessary—

* What is meant are the parts of the tangents that form an angle containing $G(t)$.

necessary and sufficient that, for arbitrary points

$$\tau_l \in K, \quad l = 1, 2, \dots, k, \quad k \geq 1; \quad t_l \in E, \quad l = 1, 2, \dots, s,$$

from the relations:

- a) $\sum_{l=1}^k \rho_l \Phi(\tau_l) + \sum_{l=1}^s \delta_l A(t_l) + \sum_{l=1}^r \sigma_l A_l = 0;$
 b) $k + s + r \leq n;$
 c) $\beta''(t_l) \leq (<) - \arg \delta_l \leq (<) \beta'(t_l), \quad l = 1, 2, \dots, s,$ if $G(t_l)$ is an unbounded domain,

it followed that

$$\rho_1 = \dots = \rho_k = \delta_1 = \dots = \delta_s = \sigma_1 = \dots = \sigma_r = 0.$$

In condition c) the sign $\leq (<)$ is used if the common part of the corresponding tangent to the domain $G(t)$ at the point $z = \infty$ and the boundary of $G(t)$ contains a certain segment (is the empty set).

If the families $G(t) \in H_{\bar{G}}$ consist of bounded domains, then item c) is superfluous. In this case the indicated necessary and sufficient condition becomes the requirement of linear independence of any system of $k + s + r$ vectors of the form

$$\Phi(\tau_l), \quad l = 1, 2, \dots, k; \quad A(t_l), \quad l = 1, 2, \dots, s; \quad A_l, \quad l = 1, 2, \dots, r;$$

$$k \geq 1, \quad k + s + r \leq n.$$

If the additional restrictions (1), (2) are absent, then Theorem 2 becomes the well-known generalization of A. Haar' s theorem due to A. N. Kolmogorov ⁽¹⁾.

4. Let us note that condition (R), under the assumption of which Theorems 1 and 2 have been proved, is essential. As examples show, violation of this condition, generally speaking, leads to the assertions of these theorems ceasing to be true. In particular, when condition (R) is violated, the Chebyshev problem may have no finite characteristic system at all. If the set E consists of a finite number of points (there are finitely many restrictions (1)), then Theorems 1 and 2 also hold in the case where condition (R) is violated. In particular, they are always valid when there are only equality restrictions (2).

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REFERENCES

¹ A. N. Kolmogorov, UMN, No. 1, 216 (1949). ² E. Ya. Remez, DAN, 77, No. 6, 965 (1951). ³ V. S. Vidensk

Note: Figure translations are in progress. See original paper for figures.

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