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L. I. KAMYNIN

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Abstract

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L. I. KAMYNIN

ON THE STABILITY OF DIFFERENCE PARABOLIC EQUATIONS

(Presented by Academician S. L. Sobolev on 16 IX 1960)

This note considers the question of solving, by the method of lines and by the method of finite differences, the first boundary-value problem for the parabolic equation

$$Lu \equiv \frac{\partial u}{\partial t} - a(x,t)\frac{\partial^2 u}{\partial x^2} - b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = F\left(x, t, u\frac{\partial u}{\partial x}\right); \tag{1}$$

$$u(x,0) = \varphi(x), \qquad 0 \le x \le l; \tag{2}$$

$$u(0,t) = u(l,t) = 0, \qquad 0 \le t \le T; \tag{3}$$

$$a(x,t) \ge a_0 > 0, \qquad c(x,t) > 0,$$

in the rectangular domain $\bar{G}$ $(0 \le x \le l,\ 0 \le t \le T)$. The well-known investigations of Lax and Richtmyer ([1,2]) make it possible to reduce the question of convergence of the method to the study of the stability of the corresponding differential-difference and finite-difference equations. In this note, certain energy inequalities for solutions of difference equations are indicated, and with their help the unconditional stability (in the sense of John ([3])) of these equations is established.

Let, in (1), the functions a, b, c, $\partial a/\partial x$, and $\partial a/\partial t$ be continuous in $\bar{G}$, and let

$$|a| \le A_1, \quad |b| \le A_2, \quad |c| \le A_3, \quad |\partial a/\partial x| \le A_4, \quad |\partial a/\partial t| \le A_5.$$

Method of lines. On the set

$$D_h: \quad \{(x,t) \in G,\ 0 \le t \le T,\ x = nh,\ n = 1, 2, \dots, N-1,\ Nh = l\}$$

consider the differential-difference operator

$$\bar{R}_h u(x,t) \equiv \frac{\partial u(x,t)}{\partial t} - a(x,t) u_{\bar{x}x}(x,t) - b(x,t) u_{\bar{x}}(x,t) + c(x,t) u(x,t),$$

where the difference quotients with forward and backward step are defined as usual:

$$u_{\bar{x}} = \frac{u(x,t) - u(x-h,t)}{h}, \qquad u_x = \frac{u(x+h,t) - u(x,t)}{h}, \qquad u_{\mathring{x}} = \frac{1}{2}(u_x + u_{\bar{x}}).$$

Let

$$\|u\|_t^2 = h \sum_{n=1}^{N} u^2(nh,t), \qquad \|\varphi\|_0^2 = h \sum_{n=1}^{N} \varphi^2(nh).$$

If the continuous function $F(x,t,y,z)$ has uniformly bounded derivatives $\partial F/\partial y$ and $\partial F/\partial z$ in $\overline{G}$,

$$\left|\frac{\partial F}{\partial y}\right| \le B_1, \qquad \left|\frac{\partial F}{\partial z}\right| \le B_2, \tag{4}$$

then for the solution of the quasilinear differential-difference equation on D_h

$$\overline{R}_h u(x,t) = F\left(x,t,u(x,t),u_{\bar{x}}(x,t)\right) \tag{5}$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad x = nh, \quad n = 0,1,2,\dots,N, \tag{6}$$

and the boundary condition (3), the energy inequality holds

$$\|u\|_t^2 + \|u_{\bar{x}}\|_t^2 \le C_1 \left(\|\varphi\|_0^2 + \|\varphi_{\bar{x}}\|_0^2 + \int_0^t \|F(x,\tau,0,0)\|_\tau^2 \, d\tau \right).$$

Definition 1 (cf. John [3]). The nonlinear differential-difference equation (5) is called **stable with respect to a given solution** $u(x,t)$ of the problem (5), (6), (3) if, for fixed T and for the chosen solution, there exists a constant B, independent of h, such that for any t_0, t $(0 \le t_0 < t \le T)$ and h on D_h, the inequality

$$\|u\|_t^2 + \|u_{\bar{x}}\|_t^2 \le B \left(\|u\|_{t_0}^2 + \|u_{\bar{x}}\|_{t_0}^2 \right). \tag{7}$$

holds.

Theorem 1. Let $F(x,t,y,z)$ satisfy (4). Then:

a) the quasilinear differential-difference equation (5) is stable for any solution of the problem (5), (6), (3);

b) if F has continuous $\partial^2 F/\partial y^2$, $\partial^2 F/\partial y\,\partial z$, $\partial^2 F/\partial z^2$, then there exists a unique solution of the problem (5), (6), (3);

c) if $u(x,t)$ is a solution of (1)–(3), and $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ are uniformly continuous on $\overline{G}$, then

$$\lim_{h\to 0} \sup_{(x,t)\in\overline{D}_h} |u(x,t)-u(x,t;h)| = 0, \tag{8}$$

where $u(x,t;h)$ is the solution of the problem (5), (6), (3);

d) if, in c), $u(x,t)$ has uniformly continuous $\partial^2 u/\partial t^2$, $\partial^3 u/\partial x^3$, $\partial^4 u/\partial x^4$ on $\overline{G}$, then

$$\sup_{(x,t)\in\overline{D}_h} |u(x,t)-u(x,t;h)| = O(h^2). \tag{9}$$

Let now $F(x,t,y,z)$ satisfy the growth restriction in y and z

$$|F(x,t,y,z)| \le f(x,t) + K\left(|y|+|z|\right)^{\alpha}, \tag{10}$$

where $K>0$ and $\alpha>0$ are constants. If $\varphi(x)$ and $F(x,t,y,z)$ are such that

$$2C_1(\alpha-1)2^{1+\alpha}\left(\|\varphi\|_0^2 + \|\varphi_{\bar{x}}\|_0^2 + \int_0^T \|f\|_t^2\,dt\right)^{\alpha-1} K^2 T < 1, \tag{11}$$

then for the solution of the nonlinear differential-difference equation (5), the energy inequality holds

$$\|u\|_t^2 + \|u_x\|_t^2 \le P\left[1-(\alpha-1)P^{\alpha-1}2^{1+\alpha}K^2 C_1 t\right]^{\frac{1}{1-\alpha}}, \tag{12}$$

where

$$P = 2C_1\left(\|\varphi\|_0^2 + \|\varphi_x\|_0^2 + \int_0^T \|f\|_t^2\,dt\right).$$

Theorem 2. If $\varphi(x)$ and $F(x,t,y,z)$ satisfy (12), then the nonlinear differential-difference equation (5) is stable for any solution of problem (5), (6), (3).

The finite-difference method. On the grid

$$G_h : \{(x,t) \in G,\ x = nh,\ n = 1,2,\dots,N-1,\ t = mk,\ m = 1,2,\dots,M\}$$

($l = Nh,\ T = Mk,\ k/h^2 = \lambda$ is constant) consider the inhomogeneous difference equation

$$R_{hu}(x,t) \equiv u_{\bar t}(x,t) - a(x,t)u_{\bar x x}(x,t) - b(x,t)u_{\bar x}(x,t) + \\ + c(x,t)u(x,t) = f(x,t) \tag{13}$$

with conditions (6) and

$$u(0,t) = u(l,t) = 0, \qquad t = mk, \quad m = 0,1,2,\dots,M. \tag{14}$$

For the solution of problem (13), (6), (14), for sufficiently small k, the following energy inequality holds (cf. Lees [4]):

$$\|u\|_t^2 + \|u_{\bar x}\|_t^2 \le C_2 \left(\|\varphi\|_0^2 + \|\varphi_{\bar x}\|_0^2 + k \sum_{s=1}^{m} \|f\|_{sk}^2 \right), \qquad t = mk.$$

Definition 2 (cf. John [3]). The linear equation (13) is called **stable** if, for the given T, there exists a constant B, independent of h, such that for every function $v(x,t)$ satisfying the homogeneous equation (13) ($f(x,t) \equiv 0$) with condition (3), for any t_0, t ($0 \le t_0 < t \le T$) and h on $\bar G_h$ ($k = \lambda h^2$), inequality (7) holds.

Theorem 3.
a) The linear difference equation (13) is unconditionally stable (i.e., for any ratio of the steps in x and t; $\lambda = k/h^2$);

b) on G_h there exists a unique solution of problem (13), (6), (14);

c) if $u(x,t)$ is a solution of (1) ($F \equiv f(x,t)$), (2), (3), and $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ are uniformly continuous on $\bar G$, then (9) holds, where D_h is replaced by G_h, and $u(x,t;h)$ is the solution of (13), (6), (14);

d) if in c), for $u(x,t)$, there exist and are uniformly continuous on $\bar G$ the derivatives $\partial^2 u/\partial t^2$, $\partial^3 u/\partial x^3$, $\partial^4 u/\partial x^4$, then (cf. Lees [4])

$$\sup_{(x,t)\in \bar G_h} |u(x,t) - u(x,t;h)| = O(k + h^2). \tag{15}$$

If $F(x,t,y,z)$ satisfies (4), then for the solution of the quasilinear difference equation on G_h

$$R_{hu}(x,t) = F\left(x, t, u(x,t), u_{\bar{x}}(x,t)\right) \tag{16}$$

with conditions (6) and (14), for sufficiently small k, the following energy inequality holds:

$$\|u\|_t^2 + \|u_{\bar{x}}\|_t^2 \leq C_3 \left(\|\varphi\|_0^2 + \|\varphi_{\bar{x}}\|_0^2 + k \sum_{s=1}^{m} \|F(x,t,0,0)\|_{sk}^2 \right).$$

Theorem 4. Let $F(x,t,y,z)$ satisfy (4).

Then:

a) the quasilinear difference equation (16) is unconditionally stable for any solution of the problem (16), (6), (13) (in the sense of Definition 1);

b) if $F(x,t,y,z)$ has continuous $\partial^2 F/\partial y^2$, $\partial^2 F/\partial y\,\partial z$, $\partial^2 F/\partial z^2$, then there exists a unique solution of the problem (16), (6), (14).

Assertions c) and d) of Theorem 1 hold with D_h replaced by G_h and (9) by (15).

Finally, suppose that $F(x,t,y,z)$ satisfies (10). If $\varphi(x)$ and $F(x,t,y,z)$ are such that

$$2C_2(\alpha-1)2^{1+\alpha} \left(\|\varphi\|_0^2 + \|\varphi_{\bar{x}}\|_0^2 + k \sum_{m=1}^{M} \|f\|_{mk}^2 \right)^{\alpha-1} K^2 T < 1, \tag{17}$$

then for the nonlinear difference equation

$$R_h u(x,t) = F\left(x, t, u(x,t-k), u_{\bar{x}}(x,t-k)\right) \tag{18}$$

for sufficiently small k the energy inequality holds

$$\|u\|_t^2 + \|u_{\bar{x}}\|_t^2 \leq P\left[1 - (\alpha-1)P^{\alpha-1}2^{1+\alpha}K^2 C_2(t-k)\right]^{\frac{1}{1-\alpha}},$$

where

$$P = 2C_2 \left(\|\varphi\|_0^2 + \|\varphi_{\bar{x}}\|_0^2 + k \sum_{m=1}^{M} \|f\|_{mk}^2 \right).$$

Theorem 5. If $\varphi(x)$ and $F(x,t,y,z)$ satisfy (17), then the nonlinear difference equation (18) is unconditionally stable (in the sense of Definition 1) for any solution of the problem (18), (6), (14).

Theorems 1–5 are proved by means of the indicated energy inequalities, obtained by applying difference analogues of Gronwall-type lemmas (see ([7], [4], [5])), the Schauder principle, and the embedding theorems of S. L. Sobolev ([6]). (C_1, C_2, C_3 are constants depending only on a_0, T and A_i, $i = 1, 2, 3, 4, 5$.)

All the arguments remain valid also for nonhomogeneous boundary conditions

$$u(0, t) = \psi_1(t), \qquad u(l, t) = \psi_2(t),$$

if one observes that the substitution

$$v(x, t) = u(x, t) - \left(1 - \frac{x}{l}\right) \psi_1(t) - \frac{x}{l} \psi_2(t)$$

reduces the problem (1), (2) to homogeneous boundary conditions (3).

For a hyperbolic equation of the form

$$\frac{\partial^2 u}{\partial t^2} - a(x, t) \frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right)$$

analogous results were obtained by M. Lees ([5]).

Moscow State University
named after M. V. Lomonosov

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