MATHEMATICS

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1961

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Abstract

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K-REGULAR CONES

(Presented by Academician P. S. Aleksandrov, October 1, 1960)

In applications of the theory of spaces with a cone to nonlinear problems, an important role is played by the possibility of passing to the limit in monotone bounded sequences. M. A. Krasnosel' skii proposed the following definitions.

Let two cones K_0 and K, where $K_0 \subset K$, be given in a real Banach space E. A semi-ordering in E is introduced by means of the "wide" cone K: $x \leq y$, if $y - x \in K$.

The cone K_0 is called K-regular if every monotone increasing and bounded (all with respect to the cone K) sequence of elements of the "narrow" cone K_0 converges in the space E, i.e., if from

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \quad (x_n \in K_0,\ n = 1, 2, ...); \tag{1}$$

$$x_n \leq u_0 \in K_0 \quad (n = 1, 2, ...) \tag{2}$$

it follows that $\|x_n - x_{n+p}\| \to 0$ as $n \to \infty$.

The cone K_0 is called completely K-regular if every monotone (with respect to the cone K) and norm-bounded sequence of elements of K_0 converges, i.e., if from (1) and

$$\|x_n\| \leq M \quad (n = 1, 2, ...) \tag{3}$$

it follows that the sequence x_n is fundamental.

For the case $K = K_0$ (in this case K-regularity and complete K-regularity are called, respectively, regularity and complete regularity), the connection between the notions of regularity, complete regularity, and normality in the sense of M. G. Krein ([2]) of a cone was studied in ([1]). In particular, it was shown that completely regular cones are regular, and regular cones are normal. There are examples ([1]) showing that not every normal cone is regular and not every regular cone is completely regular.

In the present article the case $K_0 \neq K$ is considered.

1. It is obvious that regularity and complete regularity of the "wide" cone K ensure, respectively, K-regularity and complete K-regularity of the cone K_0. It is also obvious that every K-regular (completely K-regular) cone K_0 has the property of regularity (complete regularity). However, one can indicate examples of completely regular cones K_0 which do not have the property of K-regularity with respect to some cone K (see item 2).

Let us also note that complete regularity of the cone K_0, together with K-regularity of the cone K_0, does not ensure regularity of the cone K. The corresponding example will be given below (item 2).

2. According to M. G. Krein ([2]), a cone K is called normal if there exists a $\delta > 0$ such that, for any vectors $e_1, e_2 \in K$ ($\|e_1\| = \|e_2\| = 1$), the inequality $\|e_1 + e_2\| \geq \delta$ is satisfied. As is known ([3]), normality of a cone is equivalent to the existence of such an M that from $\theta \leq x \leq y$ there follows the inequality $\|x\| \leq M\|y\|$. In connection with this, the following definition seems natural to us.

The cone K_0 is called K-normal if, for all elements $x, y \in K_0$, one can specify an N such that from $\theta \leqslant x \leqslant y$ there follows the inequality $\|x\| \leqslant N\|y\|$.

Below, $K_0\langle v, w\rangle$ denotes the set of those elements $x \in K_0$ for which $v \leqslant x \leqslant w$.

Theorem 1. (I. A. Bakhtin ([4])). *In order that the cone K_0 be K-normal, it is necessary and sufficient that all sets $K_0\langle\theta, u_0\rangle$ ($u_0 \in K_0$) be bounded.*

From the definition of K-normality it follows that every K-normal cone K_0 is normal. The converse assertion is false.

Consider, for example, in the space l_2 the set K_0 of numerical sequences $x = (x_1, x_2, \dots, x_n, \dots)$ for which $x_1 \geqslant 0$ and $0 \leqslant x_n \leqslant x_1$ ($n = 2, 3, \dots$), and as the set K take all elements $y = (y_1, y_2, \dots, y_n, \dots) \in l_2$ for which $y_1 \geqslant 0$, $-\infty < y_i \leqslant y_1$ ($i = 2, 3, \dots$). It is not hard to prove that K_0 and K are cones in l_2, and moreover $K_0 \subset K$.

The cone K_0 is part of the perfectly regular cone of nonnegative sequences from l_2, and therefore K_0 is perfectly regular and hence also normal. The elements $x_1 = (1, 0, 0, \dots)$, $x_2 = (1, 1, 0, \dots)$, $x_3 = (1, 1, 1, \dots), \dots$ belong to K_0 and are bounded above (with respect to the cone K) by the element $u = (1, 0, 0, \dots)$. However, $\|x_n\| = \sqrt{n} = \sqrt{n}\|u\|$, and the inequality $\|x_n\| \leqslant N\|u\|$ is not satisfied for any N. Thus, K_0 does not possess the property of K-normality.

It is obvious that if the "wide" cone K is normal, then the cone K_0 will be K-normal. However, from the K-normality of the cone K_0 there does not always follow the normality of the "wide" cone. For example, if the cone K_0 is finite-dimensional, then it will be K-normal and even K-regular with respect to any cone K.

Theorem 2*. *Every K-regular cone K_0 is K-normal.*

Suppose the contrary. Then there exist sequences u_n and x_n (both from K_0) such that

$$x_n \leqslant u_n \qquad (n = 1, 2, \dots) \tag{4}$$

and at the same time

$$\|x_n\| \geqslant n^2 \|u_n\|. \tag{5}$$

Consider the series $\sum_{n=1}^{\infty} \frac{u_n}{\|x_n\|}$. In view of (5), this series converges. Denote its sum by v_0. Obviously, $v_0 \in K_0$. From inequality (4) it follows that

$$\frac{x_n}{\|x_n\|} \leqslant \frac{u_n}{\|x_n\|},$$

and therefore

$$\sum_{n=1}^{\infty} \frac{x_n}{\|x_n\|} \leqslant v_0.$$

Consider the sequence

$$y_n = \sum_{k=1}^{n} \frac{x_k}{\|x_k\|}.$$

Obviously, $y_n \in K_0$ $(n = 1, 2, \dots)$, and

$$y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \leqslant \cdots \leqslant v_0.$$

Therefore, by virtue of the K-regularity of the cone K_0, y_n converges strongly, which is impossible, since

$$\|y_{n+1} - y_n\| = 1 \qquad (n = 1, 2, \dots).$$

The contradiction obtained proves the theorem.

From the theorem just proved it follows that the cone K_0 considered above is not K-regular.

3. As was already noted above, from the complete regularity of the cone K there follows its regularity. For arbitrary completely K-regular cones it has not been possible to prove their K-regularity, nor even K-normality. We have the following weaker assertions.

* M. A. Krasnosel' skii has another proof of this theorem.

Theorem 3. If the cone K_0 is completely K-regular and K-normal, then it is K-regular.

Theorem 4. If the cone K_0 is completely K-regular, then it is normal.

Theorem 5. Suppose that for every monotone norm-bounded sequence $x_n \in K$:

$$x_1 \leq x_2 \leq x_3 \leq \cdots$$

one can indicate an element u_0 such that $x_n + u_0 \in K_0$ $(n = 1, 2, ...)$. Then the complete K-regularity of the cone K_0 implies its K-regularity.

4. Let us consider the question of under what additional conditions complete K-regularity follows from K-regularity.

Theorem 6. Let the K-regular cone K_0 be solid. Then K_0 is a completely K-regular cone.

Theorem 7. A K-regular cone K_0 is completely K-regular if the cone K is normal and the space E is weakly complete.

5. We now consider the differential equation

$$\frac{dx}{dt} = f(t, x) \tag{6}$$

in a Banach space E, semi-ordered by means of a certain cone K. The operator $f(t, x)$ is assumed to be continuous in the totality of its variables. As Bourbaki showed ([5]), the existence theorem does not follow from the continuity of $f(t, x)$ (i.e., Peano' s theorem in its general form is false for equations in Banach spaces). In this connection there arises the question of additional properties which the operator $f(t, x)$ must possess in order for an existence theorem to be valid for equation (6). It turns out that a number of sufficient conditions can be formulated in terms of spaces with a cone, if one uses the notions of regularity or complete regularity of a cone.

As above, K_0 and K are cones in E, with $K_0 \subset K$. We shall assume that the operator $f(t, x)$ is defined and continuous on the topological product of the interval $[0, a]$ and the set $K_0\langle\theta, v_0\rangle$. Let $f(t, x) \in K_0$ and

$$f(t, x) \leq b(x)v_0$$

for $t \in [0, a]$, $x \in K_0\langle\theta, v_0\rangle$.

Theorem 8. Suppose that the operator $f(t, x)$, for $t \in [0, a]$, is monotone on $K_0\langle\theta, v_0\rangle$: from $x \leq y$ it follows that $f(t, x) \leq f(t, y)$. Suppose that the cone K_0 has the property of K-regularity. Then equation (6) has a solution $x(t)$, defined on some interval $[0, \delta] \subset [0, a]$, and satisfying the initial condition $x(0) = x_0$, where $x_0 \in K\langle\theta, \gamma v_0\rangle$ $(\gamma < 1)$.

The solution $x(t)$ may be obtained by the method of successive approximations:

$$x_{n+1}(t) = x_0 + \int_0^t f[s, x_n(s)]\, ds \quad (t \in [0, \delta],\ n = 0, 1, 2, ...), \qquad x_0(t) \equiv x_0.$$

Theorem 8 remains valid if, instead of requiring K-regularity of the cone K_0, one requires its complete K-regularity, and additionally requires of the operator $f(t, x)$ that the inequality

$$\sup \|f(t,x)\| < \infty \quad (t \in [0,a],\ x \in K_0\langle\theta, v_0\rangle)$$

be satisfied.

By the usual schemes ([6]), Theorem 8 can be illustrated by applications to integro-differential equations.

The author expresses sincere gratitude to M. A. Krasnosel' skii for posing the problems and for advice.

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Received
30 IX 1960

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