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MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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ON A PROCESS OF SUCCESSIVE APPROXIMATIONS AND ITS APPLICATION TO THE SOLUTION OF FUNCTIONAL EQUATIONS WITH NONLINEAR OPERATORS OF MONOTONE TYPE

(Presented by Academician A. N. Kolmogorov on 1 II 1961)

Let a functional equation be given

$$x = Ax, \quad (1)$$

where A is a nonlinear operator of monotone type mapping the given semi-ordered Banach space into itself. We shall show that the solution of this functional equation can in many cases be obtained by the method of successive approximations, using the recurrence relation

$$x_i = x_{i-1} + \alpha(x_{i-1} - Ax_{i-1}), \quad (2)$$

where α is chosen from the conditions for convergence of the process. By a similar relation A. A. Kurdyumov ⁽¹⁾ obtained the solution of certain Fredholm integral equations, and Richardson ⁽²⁾ obtained systems of algebraic equations with a symmetric positive definite matrix. In a number of works L. V. Kantorovich and his followers ⁽³⁾, by formulas of a similar type, generalized the application of Newton's method to the solution of certain functional equations of the form $P(x) = 0$, taking as the multiplier α expressions of the form $[P'(x)]^{-1}$. In the cases considered here, α is a numerical coefficient, which in practical computation in many cases proves to be more convenient.

We shall say that the operator A is monotonically increasing or monotonically decreasing if, for two elements $x_1 < x_2 \in X$, respectively the relation $Ax_1 < Ax_2$ or $Ax_1 > Ax_2$ holds. We shall say that an element $x \in X$ is positive if all its components are positive.

Using the analogue of Lagrange's formula

$$Ax_1 = Ax_0 + A'_{\bar{x}}(x_1 - x_0), \quad \text{where } \bar{x} = x_0 + \theta(x_1 - x_0) \quad (0 \leq \theta \leq 1),$$

it is easy to prove the following preliminary theorem:

Theorem 1. *If X is a semi-ordered Banach space, and Ax is a monotone operation mapping the space X into itself, admitting a first derivative $A'_x \in X$ in the sense of Fréchet, and, for increasing operators, satisfying in addition the condition $A'_x > I$ or $A'_x < I$, where I is the identity operator in X , and there exists an element x^* for which $Ax^* = x^*$, then this element is unique and is bounded by elements $x_1, x_2 \in X$, for which $Ax_1 - x_1$ and $Ax_2 - x_2$ have different signs.*

Remark. For decreasing operators the last condition will be satisfied if one takes $x_2 = Ax_1$, where x_1 is an arbitrary element such that $x_1 > x^*$ or $x_1 < x^*$, and Ax_1 exists.

For some cases of convergence of the process (2), the following two theorems hold:

Theorem 2. *If the operator A of the functional equation (1), given in the partially ordered space C with the Chebyshev metric, is an operator of monotone type and admits a first derivative in the sense of Fréchet satisfying the condition $mI \leq A'_x \leq MI$ for every element $x \in C$ bounded by the elements x_1 and x_2 defined according to Theorem 1, then for all values of the coefficients m and M for decreasing operators, and for the values $m > 1$ or $M < 1$ for increasing operators, there exists a solution x^* of equation (1). The successive approximations (2) converge to it if one takes $\alpha = \frac{1}{M_{av} - 1}$, where $M_{av} = \frac{1}{2}(M + m)$. The rate of convergence may be estimated by the inequality*

$$\|x^* - x_n\| \leq \frac{|\varepsilon|^n}{1 - |\varepsilon|} \eta_0,$$

where

$$\eta_0 \geq \|x_1 - x_0\|, \quad \varepsilon = \frac{M - m}{M + m - 2}.$$

Remark. The coefficient α can also be expressed with the aid of the Lipschitz constant K . Then $\alpha = \frac{2}{\pm K - 2}$, where the plus or minus sign is taken, respectively, for increasing and decreasing operators.

Theorem 3. *If the operator A of the functional equation (1), given in the partially ordered Hilbert space L_2 , is continuous and monotonically decreasing, then there exists a solution x^* of equation (1). The successive approximations*

(2) converge to it if one takes $\alpha = -\frac{1}{1+K^2}$, where K is the Lipschitz constant of the operator A for the elements of the interval containing x^* . The rate of convergence may be estimated by the inequality

$$\|x^* - x_n\| \leq \frac{\mu^n}{1-\mu} \eta_0,$$

where

$$\eta_0 \geq \|x_1 - x_0\|, \quad \mu = \sqrt{1-\alpha}.$$

In the proof of the last theorem one uses the inequality $\| |x| - |y| \| \leq \sqrt{\|x\|^2 + \|y\|^2}$, which holds in the partially ordered space L_2 .

The process of successive approximations set forth above is convenient both for finding approximate solutions and for the theoretical investigation of a wide class of functional equations.

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CITED LITERATURE

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2. I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, 2, Moscow, 1959.
3. L. V. Kantorovich, *Uspekhi Mat. Nauk*, 11, no. 6, 99 (1956).

Note: Figure translations are in progress. See original paper for figures.

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