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Abstract

Full Text

MATHEMATICS

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ON PROPERTIES OF A TWO-DIMENSIONAL DENSITY DEFINING A STATIONARY MARKOV PROCESS

(Presented by Academician S. N. Bernstein on 10 X 1960)

1. As is known ⁽¹⁾, a symmetric density of a two-dimensional distribution $p(t, x, y)$ with continuous parameter t defines a stationary Markov process only in the case when it is the sum of the series

$$p(t, x, y) = p(x)p(y) \left[1 + \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x) \varphi_k(y) \right], \quad (1)$$

where

$$p(x) = \int_a^b p(t, x, y) dy$$

is the a priori density of x ; $\varphi_k(x)$, $k = 1, 2, \dots$, are the eigenfunctions of the kernel

$$\frac{p(t, x, y)}{\sqrt{p(x)p(y)}},$$

orthogonal and normal on the main interval $[a, b]$ with weight $p(x)$, and the numbers λ_k , $k = 1, 2, \dots$, satisfy the conditions $0 < \lambda_1 \leq \lambda_2 \leq \dots$.

For the validity of the expansion (1), for $t > 0$, it is sufficient to impose on $p(t, x, y)$ the restriction

$$\iint_{\Omega} \frac{p^2(t, x, y)}{p(x)p(y)} dx dy < \infty, \quad (2)$$

where

$$\Omega = \left[\begin{array}{l} a \leq x \leq b \\ a \leq y \leq b \end{array} \right]$$

is a square domain, which may also be infinite.

2. If $p(x)$ and $\varphi_k(x)$, $k = 1, 2, \dots$, are continuous in $[a, b]$, then in order that the sum of the series (1) be a probability distribution density, the infinity of this series is essential, i.e., the nondegeneracy of the kernel

$$\frac{p(t, x, y)}{\sqrt{p(x)p(y)}}.$$

Theorem 1. If $p(x)$ and $\varphi_k(x)$, $k = 1, 2, \dots$, are continuous in $[a, b]$, and $p(x)$ does not vanish on this interval, then the finite sum

$$F(t, x, y) = p(x)p(y) \left[1 + \sum_{k=1}^n e^{-\lambda_k t} \varphi_k(x) \varphi_k(y) \right], \quad (3)$$

where $n \geq 1$, changes sign in the domain Ω for sufficiently small $t > 0$.

Proof. Suppose the contrary: let $F(t, x, y) \geq 0$ for all $t \geq 0$ and all $a \leq x, y \leq b$. Since $p(x) > 0$ on the interval $[a, b]$, it follows that $F(t, x, x) > 0$ for any $t \geq 0$; moreover, $F(t, x, y)$ is continuous for any x, y in Ω and any $t \geq 0$, as a finite sum of continuous functions.

Let $F(0, x, y) \geq 0$; then the kernel

$$\frac{F(0, x, y)}{\sqrt{p(x)p(y)}}$$

satisfies all conditions of a known theorem of Jentzsch⁽²⁾, i.e., it must have a unique positive eigenfunction (in our case this is $\sqrt{p(x)}$), belonging to the first simple characteristic number. But the first characteristic number of the kernel $F(0, x, y)/\sqrt{p(x)p(y)}$ has ones, and it is not simple, since its multiplicity is $n + 1$ (for at $t = 0$ all characteristic numbers $e^{-\lambda_k t} = 1$, $k = 1, 2, \dots, n$). The contradiction obtained shows that $F(0, x, y)$ changes sign in Ω , and then, by continuity, $F(t, x, y)$ will also change sign in Ω for sufficiently small positive t , as was required to prove.

Thus, if we wish to construct densities defining a stationary Markov process by means of the expansion (1), constructed from continuous $p(x)$ and $\varphi_k(x)$, we must not consider finite sums.

3. If the continuity of $\varphi_k(x)$ is abandoned, then, as will be shown, finite sums of the form (3) can define a density, but the corresponding process will be discontinuous.

Theorem 2. *There exist (discontinuous) stationary Markov processes with any a priori density $p(x)$ and discontinuous $\varphi_k(x)$, whose densities are representable by finite sums of the form (3).*

Proof. For the proof it is enough to give an example of such a process for $n = 1$. Let $p(x)$ be an arbitrary positive function such that $\int_a^b p(x) dx = 1$, and let $\lambda_1 > 0$ be an arbitrary positive number. Define the positive constant α by the condition

$$\alpha^2 = \int_a^{x_0} p(x) dx / \int_{x_0}^b p(x) dx, \quad (4)$$

where the fixed x_0 satisfies the inequalities $a < x_0 < b$.

Define the step function $\varphi_1(x)$ by the equalities

$$\varphi_1(x) = \begin{cases} -\frac{1}{\alpha}, & a \leq x < x_0, \\ \alpha, & x_0 \leq x \leq b, \end{cases} \quad (5)$$

and consider the symmetric function

$$p(t, x, y) = p(x)p(y)\{1 + e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y)\} = \begin{cases} p(x)p(y) \left[1 + \frac{1}{\alpha^2}e^{-\lambda_1 t}\right], & a \leq x, y < x_0, \\ p(x)p(y) [1 + \alpha^2 e^{-\lambda_1 t}], & x_0 \leq x, y \leq b, \\ p(x)p(y) [1 - e^{-\lambda_1 t}], & a \leq x < x_0, \quad x_0 \leq y \leq b, \\ & x_0 \leq x \leq b, \quad a \leq y < x_0. \end{cases} \quad (6)$$

From (4) and (5) it follows immediately that

$$\begin{aligned} \int_a^b \varphi_1(x)p(x) dx &= -\frac{1}{\alpha} \int_a^{x_0} p(x) dx + \alpha \int_{x_0}^b p(x) dx = \\ &= \int_{x_0}^b p(x) dx \left[-\frac{1}{\alpha} \alpha^2 + \alpha\right] = 0; \\ \int_a^b \varphi_1^2(x)p(x) dx &= \frac{1}{\alpha^2} \int_a^{x_0} p(x) dx + \alpha^2 \int_{x_0}^b p(x) dx = \end{aligned}$$

$$\begin{aligned}
 &= \int_{x_0}^b p(x) dx \left[\frac{1}{\alpha^2} \alpha^2 + \alpha^2 \right] = \int_{x_0}^b p(x) dx \left[1 + \int_a^{x_0} p(x) dx / \int_{x_0}^b p(x) dx \right] = \\
 &= \int_{x_0}^b p(x) dx + \int_a^{x_0} p(x) dx = \int_a^b p(x) dx = 1.
 \end{aligned}$$

Thus, $\varphi_1(x)$, defined by condition (5), is indeed orthogonal to one and is normal with weight $p(x)$ on the main interval $[a, b]$. Consequently,

$$\int_a^b p(t, x, y) dy = p(x),$$

while the nonnegativity of $p(t, x, y)$ is obvious for any $t \geq 0$ by the very definition (6); thus, formula (6) indeed defines the density of a stationary Markov process in the domain Ω .

Remark 1. The possibility of constructing discontinuous processes of the indicated kind was brought to my attention by B. A. Sevast'yanov.

Remark 2. If only

$$\int_a^b x^2 p(x) dx - \left[\int_a^b x p(x) dx \right]^2 > 0,$$

then

$$\lim_{t \rightarrow 0} r(t) = r(0) < 1,$$

where $r(t)$ is the normalized correlation function of the process (6), i.e. the constructed process is not continuous.

Remark 3. In an analogous manner one can construct discontinuous processes with any finite number of discontinuous eigenfunctions $\varphi_1(x), \dots, \varphi_n(x)$, orthogonal and normal on the interval $[a, b]$ with any positive weight $p(x)$, and the quantity $1 - r(0) > 0$ can be made arbitrarily small.

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References

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2. R. Jentzsch, J. f. reine u. angew. Math., **141**, H. 4 (1912).

Note: Figure translations are in progress. See original paper for figures.

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