MATHEMATICS

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1961

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Abstract

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MATHEMATICS

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EXCEPTIONAL CASES OF INTEGRAL EQUATIONS OF CONVOLUTION TYPE AND EQUATIONS OF THE FIRST KIND

(Presented by Academician V. I. Smirnov, 3 X 1960)

In recent years ($^{2-8}$), substantial results have been obtained in the theory of integral equations of convolution type with two kernels

$$\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} k_1(x-t)\varphi(t)\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} k_2(x-t)\varphi(t)\,dt = f(x), \quad \text{(A)}$$

$$-\infty < x < \infty; \quad \lambda = \lambda_1 \text{ for } x > 0, \quad \lambda = \lambda_2 \text{ for } x < 0,$$

and of the paired system

$$\lambda_1\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty,$$

$$\lambda_2\varphi(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(x-t)\varphi(t)\,dt = f(x), \quad -\infty < x < 0, \quad \text{(B)}$$

However, the theory of the corresponding equations of the first kind

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} k_1(x-t)\varphi(t)\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} k_2(x-t)\varphi(t)\,dt = f(x), \quad -\infty < x < \infty; \quad (A_0)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty;$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(x-t)\varphi(t)\,dt = f(x), \quad -\infty < x < 0, \quad (B_0)$$

has remained untouched—evidently because here one has to deal with an exceptional case. The theory of equations of convolution type has hitherto been constructed for the so-called normal case, when the coefficient of the Riemann boundary-value problem corresponding to the equation is everywhere on the contour different from zero and infinity. For equations of the first kind this condition is violated at the infinitely distant point; here there occurs the complicated case of the presence of "common zeros."

The theory of integral equations of convolution type is reduced mainly to the study of the corresponding Riemann boundary-value problem; therefore, the central issue in the theory under consideration is the study of exceptional cases of the boundary-value problem. We restrict ourselves to the case in which the coefficient has a zero or a pole of integral order. What is essentially new for the theory of integral equations is the study of the behavior of their solutions at the points of common zeros of the functions $\lambda_1 + K_1(x)$ and $\lambda_2 + K_2(x)$. It is interesting that the influence of these points on the solutions of equations of types (A) and (B) is substantially different.

1. The boundary condition of the Riemann problem for the real axis is represented in the form*

$$\Phi^+(x) = \left[\prod_{i=1}^{r}(x-a_i)^{\alpha_i} p_+(x)p_-(x)\right] \Big/ \left[\prod_{j=1}^{s}(x-b_j)^{\beta_j} q_+(x)q_-(x)\right] G_1(x)\Phi^-(x) +$$

$$+ g_1(x) \Big/ \prod_{j=1}^{s}(x-b_j)^{\beta_j}, \tag{1}$$

$$-\infty < x < \infty, \qquad \sum_{i=1}^{r}\alpha_i = m; \quad \sum_{j=1}^{s}\beta_j = n.$$

Here $p_+(x), q_+(x), p_-(x), q_-(x)$ are polynomials of degrees m_+, n_+, m_-, n_-, having zeros respectively in the upper (+) and lower (−) half-planes; $G_1(x)$ is a function satisfying the Hölder condition, nowhere (including the infinitely remote point) vanishing and having zero index; $g_1(x)$ satisfies the Hölder condition. The points a_i shall be called the zeros, and the points b_j, for brevity, the poles of the coefficient $G(x)$ of the Riemann problem. By the order of a function at infinity we shall mean the exponent of the lowest power $1/z$ in the expansion of the function in a neighborhood of the infinitely remote point. A zero will correspond to positive order, and a pole to negative order. According to the representation (1), the order of the coefficient at infinity will be expressed by the formula

$$\nu = n + n_+ + n_- - m - m_+ - m_- = n - m - \varkappa + h. \tag{2}$$

The number $\varkappa = m_+ - n_+$ shall be called the index of the problem.** We shall seek the solution in the class of functions bounded on the whole contour and vanishing at infinity.

Using the methods of L. A. Chikin ([9]) for investigating exceptional cases of the Riemann problem (see also ([1]), §§ 15, 45), for the boundary problem (1) one can write the solution in the form:

1) $\nu > 0$. The coefficient of the problem $G(x)$ has a zero at infinity.

$$\Phi^+(z) = \Gamma^+(z) + \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i} p_-(z) e^{\Gamma^+(z)}}{q_-(z)} P_{\varkappa-n-1}(z),$$

$$\Phi^-(z) = \Gamma^-(z) + \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j} q_+(z) e^{\Gamma^-(z)}}{p_+(z)} P_{\varkappa-n-1}(z). \tag{3}$$

2) $\nu < 0$. $G(x)$ has a pole at infinity.

$$\Phi^+(z) = \Gamma_1^+(z) + \frac{\prod_{i=1}^{r}(z-a_i)^{\alpha_i} p_-(z) e^{\Gamma^+(z)}}{q_-(z)} P_{h-m-1}(z),$$

$$\Phi^-(z) = \Gamma_1^-(z) + \frac{\prod_{j=1}^{s}(z-b_j)^{\beta_j} q_+(z) e^{\Gamma^-(z)}}{p_+(z)} P_{h-m-1}(z). \tag{4}$$

* Such a complicated representation is caused by the necessity of taking into account the possible singularity

** On passing from the case under consideration to the normal one ($m = n = 0$, $m_+ + m_- = n_+ + n_-$) we obtai

Here

$$\Gamma(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G_1(t)}{t-z}\, dt; \qquad \Upsilon^\pm(z),\ \Upsilon_1^\pm(z)$$

are the canonical functions of the nonhomogeneous problem ((1), p. 120) with interpolation nodes a_i, b_j of multiplicities α_i, β_j; $P_k(z)$ is a polynomial of degree k. The second terms of formulas (3) and (4) are solutions of the corresponding homogeneous problem. If $k \leq 0$, one must put $P_k(z) \equiv 0$, and for $k < 0$ formulas (3) and (4) give solutions of the problem if and only if $|k|$ solvability conditions are satisfied.

If one takes into account that, according to (2), $h-m=\kappa-n-(-\nu)$, and that in the normal case the number of linearly independent solutions is equal to the index of the problem, then the result obtained can be formulated as follows:

The number of linearly independent solutions in exceptional cases decreases, in comparison with the normal case, by the total number of all poles of the coefficient (including the pole at the infinitely distant point), and does not depend on the number of its zeros.

2. With respect to the kernels and right-hand sides of all convolution equations under consideration, we make such assumptions that the coefficients of the corresponding boundary-value problems satisfy the conditions imposed in the preceding paragraph.

To reduce the equation with two kernels (A) to a Riemann boundary-value problem, we use the methods developed in (4–7). Denoting by the corresponding capital letters the Fourier transforms of the functions entering into equation (A), we obtain that the solution of the equation is reduced to the solution of the Riemann boundary-value problem equivalent to it:

$$\Phi^+(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)}\,\Phi^-(x) + \frac{F(x)}{\lambda_1 + K_1(x)}, \qquad -\infty < x < \infty. \tag{5}$$

The solution of the original equation (A) will be expressed by the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\Phi^+(t) - \Phi^-(t)] e^{-ixt}\, dt. \tag{6}$$

Suppose that the representations

$$\begin{aligned} \lambda_1 + K_1(x) &= \prod_{j=1}^{s} (x-b_j)^{\beta_j} \prod_{k=1}^{t} (x-c_k)^{\gamma_k} K_{11}(x), \\ \lambda_2 + K_2(x) &= \prod_{i=1}^{r} (x-a_i)^{\alpha_i} \prod_{k=1}^{t} (x-c_k)^{\gamma_k} K_{12}(x), \end{aligned} \qquad \sum_{k=1}^{t} \gamma_k = l; \tag{7}$$

hold. Here a_i, b_j, c_k are real numbers; $K_{11}(x) \neq 0$, $K_{12}(x) \neq 0$. It is possible that some of the common zeros c_k coincide with a_i or b_j. The latter circumstance has no influence on the investigation; therefore such points are not singled out separately. The points a_i will be zeros, and b_j poles, of the coefficient of the Riemann problem; the infinitely distant point will be ordinary. From (5) we immediately obtain that, for the problem to be solvable, $F(x)$ must vanish to orders γ_k at all points c_k $\left(F(x) = \prod_{k=1}^{t} (x-c_k)^{\gamma_k} F_1(x)\right)$. This imposes on the right-hand side of equation (A) l conditions, which must necessarily be fulfilled in order for the equation to be solvable. Using the results of the preceding paragraph, we obtain a count of the number of solutions and, possibly, of additional solvability conditions, as well as the solutions themselves in closed form. *The number of linearly independent solutions of equation (A) coincides with the above-indicated number of solutions of the boundary-value problem, while the number of solvability conditions is greater by l. The arbitrary constants entering into the general solution cannot be used to satisfy the solvability conditions.*

3. The integral equation of the first kind (A_0) can be obtained from equation (A) for $\lambda = 0$. The corresponding boundary-value problem

$$\Phi^+(x) = \frac{K_2(x)}{K_1(x)}\Phi^-(x) + \frac{F(x)}{K_1(x)}, \quad -\infty < x < \infty, \tag{8}$$

is also obtained from (5) for $\lambda = 0$. In comparison with equation (A), the peculiarity of the latter is that the infinitely distant point will be exceptional. Since $K_1(x)$, $K_2(x)$ vanish at infinity, the infinitely distant point must be included among the points c_k where common zeros occur. Taking this circumstance into account, in studying the equation of the first kind (A_0) all results obtained for the equation of the second kind (A) may be used.

4. The paired equations (B) are reduced to a Riemann boundary-value problem by the method first indicated by I. M. Rapoport ([3]) (see also ([4−6])).

In the boundary condition of the boundary-value problem equivalent to equation (B),

$$\Omega^+(x) = \frac{\lambda_2 + K_2(x)}{\lambda_1 + K_1(x)}\Omega^-(x) + \frac{\lambda_2 - \lambda_1 + K_2(x) - K_1(x)}{\lambda_1 + K_1(x)} \tag{9}$$

the unknown is an auxiliary piecewise-analytic function $\Omega(z)$, through which the solution of the original equation is expressed by the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\Omega^+(t) + F(t)}{\lambda_2 + K_2(t)} e^{-ixt}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\Omega^-(t) + F(t)}{\lambda_1 + K_1(t)} e^{-ixt}\,dt. \tag{10}$$

The study of equation (B) on the basis of the solution of the corresponding boundary-value problem (9) is carried out analogously to how this was done above for equations (A). We obtain analogous results with only one essential difference. *Common zeros of the functions $\lambda_1 + K_1(x)$, $\lambda_2 + K_2(x)$ here also entail solvability conditions, but these may be satisfied not only by imposing restrictions on the right-hand side, but also by choosing the arbitrary constants entering into the general solution. Here the common zeros reduce the number of linearly independent solutions of the homogeneous equation, and if the difference between the index of the problem and the total order of the poles of the coefficient*

$$\left(\varkappa - \sum_{j=1}^{s} \beta_j\right)$$

is not less than the total number of common zeros, then equation (A) *will be unconditionally solvable for any right-hand side.*

The equation of the first kind (B_0) leads to the boundary-value problem obtained from (9) for $\lambda_1 = \lambda_2 = 0$. Its solution and investigation may be obtained from (B) in exactly the same way as this takes place for (A) and (A_0).

The study of the integral equation with one kernel, frequently encountered in applications,

$$\lambda\varphi(x) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k(x-t)\varphi(t)\,dt = f(x), \quad 0 < x < \infty,$$

and of the corresponding equation of the first kind ($\lambda = 0$) can be obtained from the theory of equations (A), (A_0) or (B), (B_0) for $k_2(x) \equiv 0$.

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Received
28 IX 1960

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