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Abstract

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SOLUTION OF THE RIEMANN BOUNDARY-VALUE PROBLEM FOR AUTOMORPHIC FUNCTIONS IN THE CASE OF GROUPS CHARACTERIZED BY TWO INVARIANTS

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Let a functional group Γ of fractional-linear transformations $\sigma_k(z)$, $k = 1, 2, \ldots$, be given, whose fundamental domain R has a finite number of sides, and let inside R there be a line L consisting of q simple smooth nonintersecting closed contours $l_1, l_2, \ldots, l_q$. We shall call a function $\Phi(z)$ a piecewise-holomorphic automorphic function with line of discontinuities L if it satisfies the following conditions: 1) at points of the domain R not lying on L, $\Phi(z)$ is holomorphic everywhere except, possibly, for a finite number of points at which it has poles of finite order; 2) $\Phi(z)$ is invariant under the transformations of the group Γ: $\Phi[\sigma_k(z)] = \Phi(z)$; 3) all essentially singular points of $\Phi(z)$ are limit points of the group; 4) $\Phi(z)$ is continuously continuable to every point of the line L.

We consider the following boundary-value problem:

Find a piecewise-holomorphic automorphic function $\Phi(z)$ with line of discontinuities L, having at the points $z_1, z_2, \ldots, z_m$ of the domain R poles of orders not exceeding $\mu_1, \mu_2, \ldots, \mu_m$, respectively, and satisfying the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad (1)$$

where $G(t)$ and $g(t)$ are functions of class H prescribed on L, and moreover $G(t) \neq 0$ everywhere on L.

For the case when the group Γ is completely characterized by one invariant, i.e. by one simple automorphic function $f(z)$ having in the domain R a single simple pole, a complete solution of this problem and some of its applications are given in works ([1−3]).

In the present note a solution of this problem in closed form is given for the case of groups for which there is no simple automorphic function of order equal to one. In this case the group is characterized by two simple automorphic functions $f_1(z)$ and $f_2(z)$, each of which has in the domain R a finite number of poles.

On the basis of a known property of automorphic functions $((^4), p.104)$, $f_1(z)$ and $f_2(z)$ are connected by an algebraic equation

$$Q(f_1, f_2) = 0, \tag{2}$$

where Q is an irreducible polynomial. All other simple automorphic functions belonging to the group Γ will be rational functions of $f_1(z)$ and $f_2(z)$.

If the integer p is the genus of the Riemann surface of the algebraic function defined by equation (2), or, what is the same in the present case, the genus of the fundamental domain R $((^4), p.255)$, then, as is known $(^{5,6})$, $p+1$ will be the least order of rational functions of the variables f_1, f_2. Using the Weierstrass algorithm $((^6), Ch.II)$, we construct one of such functions ⋯

automorphic functions having simple poles at the $p+1$ points $\tau, a_1, a_2, \dots, a_p$ and vanishing at $z = a_0$. This rational function of $f_1(z), f_2(z), f_1(\tau), f_2(\tau)$ we shall denote, for brevity, by $H(z,\tau)$, and, for convenience, we shall assume that the points $a_1, a_2, \dots, a_p$ and a_0 do not lie on the line L.

If the parameter τ is regarded as lying on L, then for z close to τ the following local representation holds:

$$H(z,\tau) f_1'(\tau) = \frac{1}{\tau - z} + P(z,\tau), \tag{3}$$

where $P(z,\tau)$ is a function regular in both variables. For z not lying in a neighborhood of τ,

$$H(z,\tau) f_1'(\tau) + P(z,\tau). \tag{4}$$

Finally, for z in a neighborhood of the point a_ν $(\nu = 1, 2, \dots, p)$ we have

$$H(z,\tau) f_1'(\tau) = (z - a_\nu)^{-1} H_\nu(\tau) f_1'(\tau) + P(z,\tau), \tag{5}$$

where the coefficients $H_\nu(\tau) f_1'(\tau)$ $(\nu = 1, 2, \dots, p)$ are functions regular everywhere in R and complex-linearly independent among themselves $((^6), \text{ch. } IV)$.

From the listed properties of the function $H(z,\tau)$ it follows that the integral

$$F(z) = \frac{1}{2\pi i} \int_L \varphi(\tau) H(z,\tau) f_1'(\tau)\, d\tau \tag{6}$$

represents a piecewise-holomorphic automorphic function with jump line L in the sense of the definition given by us. If the density of this integral $\varphi(\tau)$ is a function of class H, then the boundary values of this integral are computed by the formulas

$$F^{\pm}(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_L \varphi(\tau)H(t,\tau)f_1'(\tau)\,d\tau, \tag{7}$$

where the integral is understood in the sense of the principal value. Integrals of the form (7) are the basic apparatus in solving the Riemann problem posed.

Passing to the solution of the problem, let us take some point t_j as the initial point for traversing the contour l_j $(j = 1, 2, \dots, q)$, and call the index of the function $G(t)$ with respect to this contour χ_j.

In constructing the canonical function of the homogeneous problem, we apply the method of estimating an integral of Cauchy type (([7]), pp. 107–110) and use the properties of single-valued transcendental functions of Weierstrass (([6]), ch. XV, XIX):

$$E(z, t_j, z_0) = \exp\left(\int_{z_0}^{t_j} H(z,\tau)f_1'(\tau)\,d\tau\right), \qquad E_\nu(z) = \exp\left(\int_{\gamma_\nu} H(z,\tau)f_1'(\tau)\,d\tau\right),$$

where z_0 is an arbitrary point of the domain R, not coinciding with a_0 and not lying on L, and γ_ν $(\nu = 1, 2, \dots, p)$ are certain closed nonintersecting contours situated in the domain R so that the matrix of periods of Abelian integrals of the first kind

$$2\omega_{k\nu} = \int_{\gamma_\nu} H_k(\tau)f_1'(\tau)\,d\tau \qquad (k, \nu = 1, 2, \dots, p) \tag{9}$$

is nonsingular. The canonical function is obtained in the form

$$X(z) = e^{\Gamma(z)} \prod_{j=1}^{q} E^{-\chi_j}(z, t_j, z_0) \prod_{\nu=1}^{p} E_\nu^{A_\nu}(z), \tag{10}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \ln G(\tau)\, H(z,\tau)f_1'(\tau)\,d\tau, \tag{11}$$

where by $\ln G(\tau)$ one means a completely definite branch of this function; the constants A_ν $(\nu = 1, 2, \dots, p)$ are the unique solution of the system of linear algebraic equations

$$2\sum_{\nu=1}^{p} \omega_{k\nu}A_\nu = B_k \qquad (k = 1, 2, \dots, p), \tag{12}$$

where B_k are completely definite constants:

$$B_k = -\frac{1}{2\pi i}\int_L \ln G(\tau)\, H_k(\tau) f_1'(\tau)\, d\tau + \sum_{j=1}^{q} \chi_j \int_{z_0}^{t_j} H_k(\tau) f_1'(\tau)\, d\tau.$$

With this choice of the constants A_ν, the function $X(z)$ will be regular at the points $a_1, a_2, \dots, a_p$. At the point z_0, $X(z)$ has order $(-\mu_0)$, where $\mu_0 = \chi_1 + \chi_2 + \dots + \chi_q$.

The general solution of the homogeneous problem for a known function $X(z)$ is found by the usual scheme. The boundary condition is written in the form $\Phi^+(t)/X^+(t) = \Phi^-(t)/X^-(t)$, whence it follows that $\Phi(z)/X(z)$ is a simple automorphic function having, for $\mu_0 > 0$, a pole of order μ_0 at the point z_0 and poles of orders $\mu_1, \mu_2, \dots, \mu_m$ at the points $z_1, z_2, \dots, z_m$. Consequently, $\Phi(z)/X(z)$ is a rational function of $f_1(z)$ and $f_2(z)$ of the form ((6), Ch. III)

$$\frac{\Phi(z)}{X(z)} = C + \sum_{j=0}^{m} \{c_{j0} H_0(z, z_j) + c_{j1} H_1(z, z_j) + \dots + c_{j,\mu_j-1} H_{\mu_j-1}(z, z_j)\}, \quad (13)$$

where $H_\beta(z, z_j)$ are the coefficients in the expansion

$$H(z,\tau) f_1'(\tau) = \sum_\beta H_\beta(z, z_j)(\tau - z_j)^\beta, \quad (14)$$

which are simple automorphic functions having simple poles at the points $a_1, a_2, \dots, a_p$ and a pole of order $\beta + 1$ for $z = z_j$; for $z = a_0$ all $H_\beta = 0$; the constants $c_{j0}, c_{j1}, \dots, c_{j,\mu_j-1}$ are connected with one another by p equations of the form

$$\sum_{j=1}^{m} \left(c_{j0} h^j_{\nu 0} + c_{j1} h^j_{\nu 1} + \dots + c_{j,\mu_j-1} h^j_{\nu,\mu_j-1} \right) = 0, \quad (15)$$

where $h^j_{\nu k}$ denotes the coefficient of $(\tau - a_j)^k$ in the expansion of $H_\nu(\tau) f_1'(\tau)$. Equations (15) are obtained by equating to zero the coefficients of $(z - a_\nu)^{-1}$ in the expansion of the right-hand side of equality (13).

From equality (13) the general solution of the homogeneous problem is obtained in the form

$$\Phi(z) = X(z) \left\{ C + \sum_{j=0}^{m} \left[c_{j0} H_0(z, z_j) + \dots + c_{j,\mu_j-1} H_{\mu_j-1}(z, z_j) \right] \right\}. \quad (16)$$

In finding solutions that vanish at the point a_0, one must put $C = 0$. Taking into account formula (16) and equations (15), and putting $\mu = \mu_0 + \mu_1 + \dots + \mu_m$, we easily obtain the following result: for $\mu > p$, the homogeneous problem has $\mu - p$ linearly independent solutions vanishing at the point a_0; for $\mu \le p$, the homogeneous problem has no nontrivial solutions.

To obtain the general solution of the nonhomogeneous problem, one must add to the particular solution of this problem

$$\Phi_1(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)} H(z,\tau) f_1'(\tau)\, d\tau. \tag{17}$$

the right-hand side of formula (16), and adjoin to the resulting formula the system (15), in which the zeros on the right-hand sides are to be replaced by the constants

$$d_\nu = -\frac{1}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)} H_\nu(\tau) f_1'(\tau)\, d\tau. \tag{18}$$

Concerning solutions that vanish at the point a_0, we obtain the following result: for $\mu > p$ the nonhomogeneous problem has $\mu - p$ linearly independent solutions; for $\mu = p$ the solution is unique; for $\mu < p$ a (unique) solution exists only if $p - \mu$ additional conditions are satisfied, which the function $g(t)$ must obey. For $\mu = 0$, for example, these conditions have the form $d_\nu = 0$ ($\nu = 1, 2, \dots, p$).

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Note: Figure translations are in progress. See original paper for figures.