MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR B. N. DELONE

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR B. N. DELONE

PROOF OF THE FUNDAMENTAL THEOREM OF THE THEORY OF STEREOHEDRA

Regular partitions of spaces have long interested mathematicians and find important applications in crystallography and in the physics of solids. The best known are the regular partitions of the sphere. They form the basis of the theory of the so-called simple forms of crystals considered in crystallography. On the theory of the regular Dirichlet partitions corresponding to the group of parallel translations of three-dimensional space is based the method, proposed by me in 1933, for reducing (regular setting) crystals, which is now accepted in international handbooks, etc.

However, up to now one of the most fundamental theorems of the theory of regular partitions had not been proved; namely, it was not known whether, as a function of the number of dimensions n, the number of topologically distinct normal regular partitions of n-dimensional Euclidean space is bounded. This question had so far been solved only for the case $n = 2$, i.e., for the plane, and there was no visible path to its solution even in the case of ordinary three-dimensional space. In the present note I give a complete solution of this question for arbitrary n.

The solution proposed here is based on a certain modification of the famous argument of Minkowski, by means of which he proved that the number of facets of an n-dimensional parallelohedron, i.e., of a convex fundamental domain of a discrete group of parallel translations of n-dimensional Euclidean space, is not greater than $2 \cdot (2^n - 1)$.

If the whole space is filled by certain bodies, which may also have common interior points, then we call this a **filling**; if, however, these bodies pairwise have no common interior points, i.e., fill the space uniquely, then we shall call their totality a **partition of the space**. A partition is called **regular** or **isohedral** if, for any two bodies of the partition, in the group of all those motions of the space (of the first kind, or of the first and second kind) that map the partition onto itself, there is at least one motion mapping the first of these bodies onto the second. The bodies themselves of a regular partition are called **stereohedra**. If the stereohedra of the partition are convex, then they are convex polyhedra, and the partition itself is called convex. If, moreover, they are adjacent along whole facets, the partition is called **normal**.

Theorem. *The number of topologically distinct regular normal partitions of n-dimensional Euclidean space is bounded as a function only of n.*

Proof. If there is a normal regular partition, then the topology of this partition is completely determined by the following data:

1°. The topology of the edge-net of the stereohedron of the partition.

2°. The topology of the incidence of the stereohedra of the partition at the vertices of the stereohedron.

3°. The adjacency symbol[*], i.e., an indication of which facets and how adjoin with the given stereohedron S of the partition those stereohedra $S_1, S_2, \ldots$ which are adjacent to it along $(n-1)$-dimensional faces. But since, when the first two topologies 1° and 2° are given, the number of different adjacency symbols is already bounded by purely combinatorial considerations, in order to prove the theorem it is enough to prove boundedness as a function only of n of the number of different topologies 1° and 2°. To prove boundedness of the number of different topologies 1°, it is plainly enough to prove boundedness, depending only on n, of the number of faces of stereohedra; and to bound the number of different topologies 2°, it is enough to prove boundedness, depending only on n, of the number of stereohedra meeting at a vertex.

The full group G of motions of the space which carry the regular partition onto itself is, by assumption, transitive with respect to the stereohedra of the partition; i.e., whatever two stereohedra of the partition are taken, the group contains at least one motion under which the first of them passes into the second. Consider that subgroup G_0 of the group G whose motions carry some fixed stereohedron S of the partition into itself; this is a certain finite group of rotations of the stereohedron S about its center of gravity A. Let it be of order l; then the number of motions from G which carry a certain given stereohedron of the partition into a certain given other stereohedron of it is equal to l.

If the group G has a subgroup G' such that its G'_0 reduces to the identity transformation alone, then the stereohedra of the partition are fundamental regions for this group G'. In this case we call the group G' a **fundamental group** of the partition under consideration. However, we do not know whether the theorem is true that every normal regular partition of n-dimensional Euclidean space has at least one fundamental group. In the case of the Euclidean 2-dimensional plane this is so, and even in this case regular partitions have, generally speaking, several different fundamental groups. However, in the case of the 2-dimensional sphere this is not so: the regular partitions of the sphere corresponding to the icosahedron and the triacontahedron have no fundamental groups.

The group G is an n-dimensional Fedorov group (i.e., n-dimensional, discrete, and having a finite fundamental domain), and therefore, by the well-known the-

---

[*] We do not know whether one can restrict oneself only to the data 1° and 2°.

orem of Schoenflies–Bieberbach, it has an n-dimensional subgroup T of parallel translations. The index h of this subgroup relative to G is the order of the finite rotation group Ψ, consisting of the rotational parts of all motions of the group G. As the center of rotations of the group Ψ one may take the point A. This group Ψ carries into itself the lattice $\{A_t\}$ of points obtained from the point A by all translations t of the group T. By Minkowski' s theorem, the number of faces of an n-dimensional parallelohedron is not greater than $2\cdot(2^n-1)$; hence it follows that the order h of the group Ψ is not greater than some number depending only on n, since the rotations of Ψ plainly carry into itself that parallelohedron which is the Dirichlet domain of the point A in the lattice $\{A_t\}$.

The group G_0 is the group of those motions of the group G which leave the point A fixed; therefore the group G_0 is a subgroup of Ψ. Let

$$\Psi = G_0 + \psi_2 G_0 + \psi_3 G_0 + \ldots + \psi_{h'} G_0 \quad (\text{where } h' = h/l)$$

be the decomposition of the group Ψ into cosets with respect to its subgroup G_0, and let $1, g_2, g_3, \ldots, g_{h'}$ be motions from G whose rotational parts are $1, \psi_2, \psi_3, \ldots, \psi_{h'}$. In this case all stereohedra of the partition split into h' nonintersecting "lattices of stereohedra," namely into such aggregates of stereohedra whose centers of gravity form the lattices $\{A_t\}, \{A_{g_2 t}\}, \ldots, \{A_{g_{h'} t}\}$, and whose rotations relative to the initial stereohedron S under consideration are $1, \psi_2, \psi_3, \ldots, \psi_{h'}$.

After these preliminary considerations we pass to the proof of the theorem.

We shall first show that the number of $(n-1)$-dimensional faces of a stereohedron is not greater than

$$2\cdot(2^n-1) + (h'-1)\cdot 2^n.$$

Indeed, consider those $(n-1)$-dimensional faces of the stereohedron S under consideration along which it is touched by stereohedra of any one and the same of the h' "lattices of stereohedra" considered above. If the number of such stereohedra were greater than 2^n, then among them there would be two, S_1 and S_3, whose corresponding coordinates of the centers of gravity, relative to the fundamental parallelepiped of the lattice to which they belong, would be congruent modulo 2; then the vector going from the center of gravity of one of these stereohedra to the center of gravity of the other would have even coordinates, i.e., would be twice an integral vector of this lattice. But then there would exist a third stereohedron S_2 such that S_2 is obtained from S_1 by the same translation t by which S_3 is obtained from S_2. Let the $(n-1)$-dimensional faces of the stereohedron S along which it is touched by the stereohedra S_1 and S_3 be b and a. Namely, the stereohedron S_1 touches S along its face b with its own face b_1, and the stereohedron S_3 touches S along its face a with its own face a_3. On the stereohedra S_1, S_2, S_3 there are faces $a_1 b_1, a_2 b_2, a_3 b_3$, obtained from b_1 and a_3

by the corresponding parallel translations. Let A and B be some interior points of the faces a and b, and let A_1B_1, A_2B_2, A_3B_3 be the corresponding interior points of the faces a_1b_1, a_2b_2, a_3b_3, with the points B and B_1, and the points A and A_3, coinciding. Join by straight-line segments the points A_1B_1, A_2B_2, A_3B_3 and the points B_1 and A_3. The following figure is obtained:

on which $A_1B_1B_3A_3$ is a parallelogram, A_2B_2 is its midline, and B_1A_3 is a diagonal. By virtue of the convexity of the stereohedra, all interior points of the segment A_2B_2 are interior points of the stereohedron S_2, and the interior points of the segment B_1A_3 are interior points of the stereohedron S. The point C of intersection of these segments, which is their common midpoint, is therefore a common interior point of the stereohedra S_2 and S, i.e., these stereohedra coincide.

Hence we conclude that if S does not belong to that "lattice of stereohedra" whose contacts along $(n-1)$-dimensional faces with the stereohedron S we are studying, then the number of stereohedra of such a lattice touching S along $(n-1)$-dimensional faces is not greater than 2^n. If, however, S belongs to this lattice, then there are not more than $2 \cdot (2^n - 1)$ of them, since in this case to each stereohedron S_1 of this lattice touching S along an $(n-1)$-dimensional face there corresponds another stereohedron S_3 of it, also touching S along an $(n-1)$-dimensional face, such that the triple S_1SS_3 is precisely the triple $S_1S_2S_3$.

Let us now prove that the number of stereohedra meeting at one vertex is not greater than $h!2^n$. Indeed, if more than 2^n stereohedra of one and the same "lattice of stereohedra" came to the vertex, then in it there would be a triple $S_1S_2S_3$ such that its stereohedra S_1 and S_3 would have a common point. But this is impossible, since if P is some $(n-1)$-dimensional plane separating S_1 and S_2—and such a plane always exists, since S_1 and S_2 are convex and have no common interior points—then the plane P', obtained from it by the translation t by which S_2 is obtained from S_1, or S_3 from S_2, separates S_2 from S_3, and therefore the stereohedra S_1 and S_3 are separated from one another by a layer ("slab") PP' of nonzero thickness. The latter is so because S_2 lies between P and P'.

Received
24 III 1961

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.