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Abstract

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On Necessary and Sufficient Conditions for the Stability of Systems of Difference Equations

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In the present note we shall indicate necessary and sufficient conditions for the stability of difference systems of the form

$$u_p^{n+1,k} = \sum_{j,l} a_{p,j}^l u_j^{n,k+l}, \tag{1}$$

$p, j = 1, 2, \dots, m;$ l is bounded, with initial data from the class $\{l_2\}$ with norm

$$\|u_p^n\| = \left\{ \sum_{k=-\infty}^{+\infty} |u_p^{n,k}|^2 \right\}^{1/2}.$$

To the difference system (1), by means of the Fourier transform, one can associate the system

$$U^n(t) = A^n(t) U^0(t), \tag{2}$$

where $t = e^{is}$.

We have formulated ([1]) a theorem on the possibility of reducing a matrix with analytic elements to triangular form by a similarity transformation with a non-singular analytic transition matrix. In connection with this theorem, several remarks should be made.

Remark 1. If at the point $s = s_0$ the eigenvalues split into groups, in each of which they are equal to one another, then in a neighborhood of $s = s_0$ the matrix can be reduced to block-triangular form with the number of blocks equal to the number of groups of eigenvalues.

Remark 2. When a block-triangular matrix is raised to a power, each block is raised to the power independently of the others.

Suppose that the eigenvalues of the matrix $A(t)$ satisfy the conditions: $|\lambda_i| < q_0 < 1$ outside some neighborhood of the point $s = 0$, while inside it the λ_i are analytic functions of the parameter s.

As we showed ([1]), the asymptotic behavior of the solution of system (1) will be determined by the matrix $A_0(t)$, whose elements coincide with $A(t)$ inside a neighborhood of $s = 0$ and are identically equal to zero outside it.

Let us suppose that in a neighborhood of the point $s = 0$ we have established a similarity transformation, as a result of which the system obtained is

$$\overline{U}^n(t) = \overline{A}_0^n(t)\overline{U}^0(t), \tag{3}$$

where $\overline{A}_0(t)$ is a triangular matrix.

Every triangular matrix can be represented in the form $D+N$, where D is diagonal and N is nilpotent. By induction the following formula can be established:

$$(D+N)^n = D^n + N^{(1)} + N^{(2)} + \dots + N^{(k-1)}, \tag{4}$$

where $N^{(p)}$ is the p-th power of the matrix N, whose elements have been multiplied by certain functions of λ_i, and k is the order of the matrix N. Let $N = \{a_{ik}\}$, where the a_{ik} lying on the main diagonal and above it are identically equal to zero. Then

$$N^{(1)} = \{a_{ik}\delta_{ik}^n\}, \qquad N^{(2)} = \left\{\sum_{i<p<k} a_{ip}a_{pk}\delta_{ipk}^n\right\}, \dots,$$

where

$$\delta_{ik}^n = \lambda_i^n/(\lambda_i - \lambda_k) + \lambda_k^n/(\lambda_k - \lambda_i),$$

$$\delta_{ipk}^n = \lambda_i^n/(\lambda_i - \lambda_p)(\lambda_i - \lambda_k) + \lambda_p^n/(\lambda_p - \lambda_i)(\lambda_p - \lambda_k) + \tag{5}$$

$$+\lambda_k^n/(\lambda_k - \lambda_p)(\lambda_k - \lambda_i).$$

We called the system (1) stable if

$$\|u_p^n\| \le K \sum_j \|u_j^0\|. \tag{6}$$

We shall now determine the conditions for stability of the difference scheme generated by system (3), and hence of scheme (1).

Theorem. In order that system (3) generate a stable difference scheme, it is necessary and sufficient that:

I. The eigenvalues of the matrix $A(t)$ in a neighborhood of $s = 0$ satisfy the inequality $|\lambda_i| \le 1$ (we shall assume that at the point $s = 0$, $|\lambda_i| = 1$ and the matrix consists of a single cell).

Taking into account that $|\lambda_i| \le 1$ and $|\lambda_i| = 1$ at $s = 0$, we obtain

$$\lambda_k = \exp\{u_k(s) + iv_k(s)\} = \alpha_0^{(k)} + \alpha_1^{(k)} s + \alpha_2^{(k)} s^2 + \cdots + \alpha_n^{(k)} s^n + \cdots,$$

$$\ln \lambda_k = i \sum_{j=0}^{2q} \alpha_j^{(k)} s^j - \beta_{2q}^{(k)} s^{2q} + \cdots, \tag{7}$$

where $\beta_{2q} \ge 0$, and the index $2q$ depends on k.

II. The elements of the matrix $\bar{A}_0$ must satisfy the following conditions:

Let λ_k and λ_l be a pair of eigenvalues and

$$\alpha_0^{(k)} = \alpha_0^{(l)}, \qquad \alpha_1^{(k)} = \alpha_1^{(l)}, \dots, \qquad \alpha_j^{(k)} = \alpha_j^{(l)}.$$

Then:

a) if $j < 2q(k)$, $j < 2q(l)$, the element a_{kl} must have at the point $s = 0$ a zero of multiplicity not less than $j + 1$;

b) if $j \ge 2q(k) = 2q(l)$, the element a_{kl} must have at the point $s = 0$ a zero of multiplicity not less than $2q$;

c) if $\lambda_k \equiv \lambda_l$ and $|\lambda_k|, |\lambda_l| \equiv 1$ throughout a neighborhood of $s = 0$, the element a_{kl} must be identically equal to zero in a neighborhood of $s = 0$.

We outline the proof of the theorem.

Sufficiency. In case a), a_{kl} has a zero of multiplicity not less than $\lambda_k - \lambda_l$, whence it is seen that $a_{kl}\delta_{kl}^n$ has no singularity, and, by item I, the norm $a_{kl}\delta_{kl}^n$ is bounded.

In case b), for sufficiently small s the inequalities

$$|\lambda_l|,\ |\lambda_k| \ge \exp(-nbs^{2q}), \qquad |\delta_{kl}^n| \le n \exp(-nbs^{2q}),$$

hold, where $b > 0$. Hence we have

$$\|a_{kl}\delta_{kl}^n\|^2 \le K_1 n^2 \int_{-\varepsilon}^{+\varepsilon} s^{4q} \exp(-nbs^{2q})\, ds \sim K_2 n^{-1/2q}.$$

We see that the norm $\|a_{kl}\delta_{kl}^n\|$ does not increase with increasing n.

The sufficiency of item c) is obvious.

The elements of the matrix $\bar{A}_0^n$ contain terms of the form

$$a_{ik}a_{kl}\dots a_{qm}\delta^n_{i,k,l,\dots,q,m}. \tag{8}$$

By induction one can establish that, when conditions of type II are fulfilled for the elements $a_{ik}, a_{kl}, \dots, a_{qm}$, the norm (8) is also bounded as $n \to \infty$.

Necessity. Suppose conditions I and II are not fulfilled; then there exist initial data such that system (1) proves unstable. Suppose, for example, that in a neighborhood of $s = 0$, $|\lambda_k| > q_0 > 1$. Then, choosing the initial data in the form $U_j^0 = 1,\ j = k;\ U_j^0 = 0,\ j \neq k$, we obtain $\|U_k^n\|^2 \geq K_3 q_0^n \to \infty$. The necessity of I is proved. Next one can obtain the necessity of conditions II for the elements $a_{12}, a_{23}, \dots, a_{k-1,k}$; from the stability conditions for these elements follows the boundedness of the norms of the terms that are added to the elements $a_{13}, a_{24}, \dots, a_{k-2,k}$, after which the necessity of conditions II for these elements is verified. In this way, successively, we exhaust all elements of the matrix $\bar{A}_0$.

Take the element a_{kl} and suppose that the condition of type c) is not fulfilled. Then $\|a_{kl}\delta_{kl}^n\| \sim K_4 n$.

Suppose the condition b) $a_{kl} = s^{2q-1}a'_{kl}$ is not fulfilled; then the estimate holds

$$\begin{aligned}\|a_{kl}\delta_{kl}^n\|^2 \sim A\int_{-\alpha}^{+\alpha} [\exp nu_k - \exp nu_l]^2 s^{4q-2k_1-2}\,ds \\ + B\int_{-\alpha}^{+\alpha} \exp n(u_k+u_l)\sin^2 n\frac{v_k - v_l}{2}\, s^{4q-2k_1-2}\,ds.\end{aligned} \tag{9}$$

Estimating the first term, we obtain

$$\int_{-\alpha}^{+\alpha} [\exp nu_k - \exp nu_l]^2 s^{2q-2k_1-2}\,ds \sim A_1 n^{\frac{1}{2q}} + A_2 n^{-\frac{k_2-1}{2q}} + A_3 n^{-\frac{2k_2-1}{2q}} + \cdots, \tag{10}$$

where $k_1 = j+1,\ k_2 \geq 1$.

Estimate the second integral:

$$\int_{-\alpha}^{+\alpha} \exp n(u_k+u_l)\sin^2 n\frac{v_k - v_l}{2}\, s^{4q-2k_1-2}\,ds \sim B_1 n^{\frac{1}{2q}} + B_2 n^{-\frac{2k_2-1}{2q}} + \cdots. \tag{11}$$

We see that $\|\delta_{kl}^n a_{kl}\|$ grows as $n^{\frac{1}{2q}}$.

Suppose now that in case a) the element a_{kl} has a zero of multiplicity j and $2q(k) \neq 2q(l)$.

Integrating by parts, estimate the first integral of type (10):

$$\int_{-\alpha}^{+\alpha} [\exp nu_k - \exp u_l]^2 s^{-2}\, ds \sim C_1 e^{-nb} + C_2 n^{\frac{1}{2p}} + C_3 n^{\frac{1}{2q}} + \cdots. \tag{12}$$

Estimating the second integral of type (11), we obtain

$$\int_{-\alpha}^{+\alpha} \exp n(u_k + u_l) \sin^2 n \frac{v_k - v_l}{2}\, s^{-2}\, ds \sim D n^{\frac{1}{j+1}}. \tag{13}$$

In the case when the system is stable by virtue of items a) and c) of the theorem, this difference system can be reduced to diagonal form.

In the book [2] sufficient conditions for stability are indicated and examples are given which show that, generally speaking, these conditions cannot be weakened. However, there exist examples of schemes that remain stable when conditions [2] are violated.

Let us try to clarify, in the light of the theorem formulated, the causes of such phenomena.

Example 1.

$$u_k^{n+1} = u_{k+1}^n + u_{k-1}^n - v_k^n, \qquad v_k^{n+1} = u_k^n. \tag{14}$$

This system corresponds to the matrix

$$\bar{A}(t) = \begin{vmatrix} t & 0 \\ 1 & \dfrac{1}{t} \end{vmatrix}. \tag{15}$$

We see that for $s = 0$, $\lambda_1 = \lambda_2 = 1$; for $s = \pi$, $\lambda_1 = \lambda_2 = -1$, i.e., one of conditions [2] is violated. For stability it is required that a_{12} have a zero of at least the same multiplicity as $\lambda_1 - \lambda_2 = 2i \sin s$ at the points $s = 0$, $s = \pi$. In the present case $a_{12} = 1$, and the necessary and sufficient conditions are violated. The order of growth of the solution is determined by the asymptotics of the integral:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\sin^2 ns}{\sin^2 s}\, ds \sim An + B + o\left(\frac{1}{n}\right). \tag{16}$$

Example 2.

$$u_k^{n+1} = \frac{1}{2}\left(u_{k+1}^n + u_{k-1}^n\right) + \frac{1}{2}\left(v_{k+1}^n - v_{k-1}^n\right),$$

$$v_k^{n+1} = \frac{1}{2}\left(u_{k+1}^n - u_{k-1}^n\right) + \frac{1}{2}\left(v_{k+1}^n + v_{k-1}^n\right). \qquad (17)$$

The matrix $\bar{A}(t)$ has the form

$$\bar{A}(t) = \begin{vmatrix} t & 0 \\ \frac{1}{2}\left(t - \frac{1}{t}\right) & \frac{1}{t} \end{vmatrix}. \qquad (18)$$

The situation is similar to Example 1; however, at the points $s = 0$, $s = \pi$, $a_{12} = i \sin s$ satisfies the stability conditions.

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2. V. S. Ryabenkii, A. F. Filippov, *On the Stability of Difference Equations*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.