R. S. ISMAGILOV

In the present note we establish one sufficient criterion for the semiboundedness of the operator

1961

SovietRxiv

Abstract

Full Text

R. S. ISMAGILOV

ON CONDITIONS FOR SEMIBOUNDEDNESS AND DISCRETENESS OF THE SPECTRUM FOR ONE-DIMENSIONAL DIFFERENTIAL OPERATORS

(Presented by Academician P. S. Aleksandrov, 20 IV 1961)

In the present note we establish one sufficient criterion for the semiboundedness of the operator

$$L = (-1)^n \frac{d^{2n}}{dx^{2n}} + q(x) \qquad (-\infty < x < \infty,\ \operatorname{Im} q(x) = 0) \tag{1}$$

and, for operators (1) satisfying this criterion, we indicate a criterion for discreteness of the spectrum. Under the additional assumption $q(x) > -c$, a similar criterion was found by A. M. Molchanov ([1]) and consists, as is known, in the fact that

$$\int_x^{x+a} q(s)\, ds \to \infty$$

as $x \to \infty$ for all $a > 0$. In ([2]) it is shown (only for $n = 1$) that this criterion remains valid if $q(x)$ is subject to the less restrictive requirement:

$$\int_t^x q(s)\, ds > -c$$

in the region $0 \le x - t \le \delta$ for some c and δ; moreover, as shown in that paper, this last condition implies the semiboundedness of the operator (for $n = 1$). Below we present another criterion for discreteness of the spectrum, valid in a considerably broader class of semibounded operators L. It turns out that in this class Molchanov' s criterion is no longer valid.

Theorem 1. *Suppose that for some $\delta > 0$*

$$\int_t^x q(s)\, ds > \alpha(x) - \beta(t) \qquad \text{in the strip } 0 \le x - t \le \delta \tag{2'}$$

and

$$\int_\Delta \alpha^2(s) + \beta^2(s)\, ds < c$$

for all intervals Δ *of length* 1 $(2'')$. *Then: a) the operator (1) is semibounded; b) for the spectrum of the operator to be discrete it is necessary and sufficient that the following condition hold: if the square* $K_a = (0 \le x \le \delta,\ a \le x+t \le a+\delta)$ *in the* (x,t)*-plane tends to infinity, preserving its size and remaining inside the strip* $0 \le x - t \le \delta$, *then the function*

$$\int_t^x q(s)\, ds,$$

considered on the squares K_a, *tends to* ∞ *in measure; i.e. for every number* A

$$\operatorname{mes} E\left\{(x,t) : \int_t^x q(s)\, ds < A\right\} \cap K_a \to 0 \qquad \text{as } |a| \to \infty. \tag{3}$$

Before proving the theorem, we establish some lemmas concerning arbitrary operators of the form (1). We first consider the case $n = 1$.

Lemma 1. *Cover the axis* $(-\infty,\infty)$ *by a finite or countable number of intervals* Δ_k *so that* Δ_k *and* Δ_{k-1} *intersect in a segment* δ_k *of length* h_i, *and* $\Delta_i \cap \Delta_j = \Lambda$ $(i - j > 1)$. *Take smooth functions* $\varphi_k(x)$ *such that*

$$\sum \varphi_k^2(x) \equiv 1$$

and $\varphi_k(x) = 0$ *for* $x \notin \Delta_k$. *Put* $y_k = y\varphi_k^2$ *and* $u_k = y\varphi_k\varphi_{k-1}$.

Then the identity holds

$$(Ly,y) = \sum (Ly_k, y_k) + 2\sum (Lu_k, u_k) - \int y^2(\varphi_k'\varphi_{k-1} - \varphi_k\varphi_{k-1}')^2\, ds. \tag{4}$$

The proof is carried out by a straightforward computation.

Let $\Delta = (a,b)$; denote $\inf \dfrac{(Ly,y)}{(y,y)}$, $y(a) = y(b) = 0$, by $\lambda(\Delta) \equiv \lambda(a,b)$.

Lemma 2. If Δ is an interval of length δ and $h < \delta$, then in Δ there exists an interval Δ' of length not exceeding h such that

$$\lambda(\Delta') \le \lambda(\Delta) + \frac{36}{h^2}. \tag{5}$$

Proof. From the definition of the number $\lambda(\Delta)$ it follows that there exists a function $y(x)$, equal to 0 outside Δ, such that

$$(Ly, y) \le \lambda(\Delta)(y, y). \tag{6}$$

Construct the intervals Δ_k, δ_k and the functions y_k as in Lemma 1; in addition, suppose that the length of Δ_k is equal to h, and the length of δ_k is equal to ${}^1/_3h$ and $|\varphi_k'| < 3/h$. Then from equality (4) we obtain

$$(Ly, y) \ge \sum (Ly_k, y_k) + 2\sum (Lu_k, u_k) - \frac{36}{h^2}(y, y). \tag{7}$$

But since $y_k y_j \equiv 0$ for $|k-j| > 1$, we have

$$(y, y) = \sum \int (y_k^2 + 2\sum y_i y_{i+1})\, ds = \sum \int (y_k^2 + 2\sum u_k^2)\, ds. \tag{8}$$

From (6), (7), and (8) it follows that

$$\sum (Ly_k, y_k) + 2\sum (Lu_k, u_k) \le \left(\lambda(\Delta) + \frac{36}{h^2}\right)\left(\sum (y_k, y_k) + 2(u_k, u_k)\right),$$

or, if $\lambda(\Delta) + 36/h^2$ is denoted by m,

$$\sum [(Ly_k, y_k) - m(y_k, y_k)] + 2\sum [(Lu_k, u_k) - m(u_k, u_k)] \le 0,$$

whence it follows that at least one of the brackets is nonpositive. But this means precisely that, for the corresponding interval τ,

$$\lambda(\tau) \le m = \lambda(\Delta) + \frac{36}{h^2}.$$

Corollary ("localization principle" ([1])). For the semiboundedness of the operator $-d^2/dx^2 + q(x)$ it is necessary and sufficient that the numbers $\lambda(\Delta)$ be uniformly bounded below for all intervals Δ of fixed length.

Lemma 3. For discreteness of the spectrum of the semibounded operator $-d^2/dx^2 + q(x)$ it is necessary and sufficient that $\lambda(\Delta) \to \infty$ when the interval Δ tends to ∞, while preserving its length.

Proof. We shall assume that $(Ly, y) > (y, y)$. Necessity follows from Rellich' s lemma ([1]).

Suppose now that $\lambda(\Delta) \to \infty$. From Lemma 2 it follows that then $\lambda(w, \infty) \to \infty$ as $w \to \infty$. Construct the intervals $\Delta_1 = (-\infty, -n+1)$, $\Delta_2 = (-n, n)$,

$\Delta_3 = (n-1, \infty)$ and the functions φ_k $(k = 1, 2, 3)$ as in Lemma 1. Let $|\varphi_k'| < 2$. From (4) it follows that

$$\sum (Ly_k, y_k) \leq (Ly, y) + 4(y, y)$$

or, since $(Ly, y) > (y, y)$,

$$\sum (Ly_k, y_k) \leq 5(Ly, y).$$

But

$$(Ly_1, y_1) + (Ly_2, y_2) > \lambda(n, \infty)((y_1, y_1) + (y_3, y_3)).$$

Therefore

$$(Ly_2, y_2) + \lambda(n, \infty)((y_1, y_1) + (y_2, y_2)) \leq 5(Ly, y). \tag{*}$$

Let n be so large that $\lambda(n, \infty) > A^2$.

Consider the set $M\{y(x) : (Ly, y) < 1\}$. From $(*)$ it follows that every $y \in M$ admits a decomposition $y = y_1 + y_2 + y_3$ and $(Ly_2, y_2) < 5$, $\|y_1\|^2 + \|y_2\|^2 < 5/A^2$, $y_i = 0$ for $x \notin \Delta_i$.

But the set $\{y : (Ly, y) < 5,\ y(x) = 0 \text{ for } x \notin \Delta_2\}$ is compact. Therefore

the set M admits a finite $\frac{\sqrt{5}}{A}$-net. M is compact by virtue of the arbitrariness of A. It follows from Rellikh' s lemma that the spectrum is discrete, as was required.

For the case $n > 1$, identity (4) from Lemma 1 is replaced by the following:

$$(Ly, y) = \sum (L' y_k, y_k) + 2 \sum (L'' u_k, u_k),$$

where $L' = L + L''$, and L'' is an operator of order $2n - 2$.

From this identity we obtain, as above, that the uniform semiboundedness from below of the operator L' on all intervals Δ of fixed length entails the semiboundedness of the operator L, and that for discreteness of the spectrum of the operator L it is sufficient (but, possibly, not necessary) that $\lambda'(\Delta) \to \infty$ as $\Delta \to \infty$, while preserving the length (here

$$\lambda'(\Delta) = \inf_y \frac{(L' y, y)}{(y, y)},$$

where y is a smooth function vanishing outside Δ).

If we denote

$$L^\varepsilon = (1-\varepsilon)(-1)^n \frac{d^{2n}}{dx^{2n}} + q(x)$$

and

$$\lambda^\varepsilon(\Delta) = \inf \frac{(L^\varepsilon y, y)}{(y,y)}$$

($y = 0$ outside Δ), then $L^\varepsilon < L' + C$ (where $C = C(\varepsilon)$). Therefore, for semiboundedness (respectively, for discreteness of the spectrum) of the operator L, it is sufficient that $\lambda^\varepsilon(\Delta) > -C$ for all Δ of equal length (respectively, that $\lambda^\varepsilon(\Delta) \to \infty$ as $\Delta \to \infty$, while preserving its length).

Proof of the theorem

I. Let first $n = 1$. Suppose that conditions $(2')$ and $(2'')$ are satisfied; denote $\min(\delta, 1/16c)$ by h. From the preceding it follows that the problem of semiboundedness and discreteness of the spectrum reduces to the following: on the interval $(0,h)$ there is given a sequence of forms

$$(L_{ny}, y) = \int_0^h ({y'}^2 + q_{ny}^2)\, ds, \qquad y(0) = y(h) = 0.$$

Under what conditions: a) $\lambda_n(0,h) > -C$ and b) $\lambda_n(0,h) \to \infty$ (here

$$\lambda_n = \inf \frac{(L_{ny}, y)}{(y,y)}, \qquad y(0) = y(h) = 0$$

)?

Let

$$Q_n(x) = \int_0^x q_n(t)\, dt, \qquad P_n(x) = \sup_{0 \le t \le x \le h} [\, Q(t) - \beta(t)\,], \qquad R_n(x) = Q_n(x) - P_n(x).$$

From $(2')$ and $(2'')$ it follows that

$$\alpha(x) \le R_n(x) \le \beta(x), \qquad \int_0^h R_n^2(x)\, dx < C.$$

If $y(0) = y(h) = 0$, then

$$\int_0^h ({y'}^2 + q_{ny}^2)\, ds = \int_0^h ({y'}^2 + P_n' y^2 + R_n' y^2)\, ds = \int_0^h ({y'}^2 + P_n' y^2 - 2R_{nyy}')\, ds.$$

It is clear that $|y(x)| \le \|y'\|\sqrt{h}$ and

$$\left|2\int R'_{nyy}\,ds\right| \le 2\|y'\|^2\|R_n\|\sqrt{h},$$

or, by virtue of the choice of h,

$$\left|2\int R'_{nyy}\,ds\right| \le \frac{1}{2}\|y'\|^2.$$

Therefore

$$\frac{1}{2}\int ({y'}^2 + P'_n y^2)\,ds \le \int ({y'}^2 + q^2_{ny})\,ds \le \frac{3}{2}\int ({y'}^2 + P'_n y^2)\,ds. \tag{9}$$

Hence it is clear that $\lambda_n(0,h) > 0$ and that the condition $\lambda_n(0,h) \to \infty$ is equivalent to the fact that

$$\inf \int ({y'}^2 + P'_n y^2)\,ds$$

$(\|y\| = 1,\ y(0) = y(h) = 0)$ tends to ∞. But since $P'_n \ge 0$, this is possible if and only if

$$\int_\Delta P'_n\,ds \to \infty$$

for every interval $\Delta \in (0,h)$; this latter condition is equivalent to the fact that

that $P_n(x) - P_n(t) \to \infty$ in the triangle $D_h = (0 \le t < x \le h)$. Taking into account that $P'_n \ge 0$, it is easy to show that this is equivalent to convergence of the sequence under consideration in measure to ∞. But since $Q_n(x) - Q_n(t) = P_n(x) - P_n(t) + R_n(x) - R_n(t)$ in D_h and $\|R_n(x) - R_n(t)\|_{L_2(D_h)} < C$, this is possible if and only if $Q_n(x) - Q_n(t) \to \infty$ in measure in D_h. It is easy to show that this is equivalent to assertion b) of Theorem 1.

II. Let $n > 1$. Since

$$L \equiv (-1)^n \frac{d^{2n}}{dx^{2n}} + q(x) = \left[(-1)^n \frac{d^{2n}}{dx^{2n}} + \frac{d^2}{dx^2}\right] + \left[-\frac{d^2}{dx^2} + q(x)\right]$$

and the first summand is an operator bounded below, it follows from the boundedness below and discreteness of the spectrum of the operator $-d^2/dx^2 + q(x)$ that L also possesses these properties. The necessity of the discreteness criterion follows from the inequality

$$\int_0^h \left(|y^{(n)}|^2 + q_{my}^2\right) ds \le \left(1 + \frac{h^{2n-1}}{n!}\right) \int_0^h \left(|y^{(n)}|^2 + P_m' y^2\right) ds$$

analogously to (9). The theorem is proved.

Let us give an example of a bounded-below operator $-d^2/dx^2 + q(x)$ which has a discrete spectrum and does not satisfy the condition of A. M. Molchanov. Take sequences of positive numbers m_n and p_n such that

$$m_n^2 p_n < c, \qquad \frac{m_{n+1}}{m_n} = \frac{1 - p_{n+1}}{p_n}, \qquad m_n \uparrow \infty, \qquad p_n \to 0 \quad (|n| \to \infty).$$

On each interval $(n-1, n)$ of the axis $(-\infty, \infty)$, construct a continuous function $Q_n(x)$, linear on the intervals $(n-1, n-p_n)$ and $(n-p_n, n)$, equal to zero at the points $n-1$ and n, and equal to m_n at the point p_n. The function $Q(x)$ obtained in this way has a piecewise-continuous derivative $q(x)$. It is not difficult to show that $q(x)$ satisfies the condition of the theorem, so that the operator $-d^2/dx^2 + q(x)$ is bounded below and has a discrete spectrum. At the same time

$$\int_n^{n+1} q(s)\, ds = Q(n+1) - Q(n) = 0$$

for integer n.

The arguments set forth above on the "localization principle" carry over almost without change to operators of the form

$$L = L_0 + q(x_1, \dots, x_n), \qquad \text{where} \qquad L_0 = (-1)^m \sum_{i=1}^{n} \frac{\partial^{2m}}{\partial x_i^{2m}}.$$

If we denote

$$L^\varepsilon = (1-\varepsilon)L_0 + q \quad \text{and} \quad \lambda^\varepsilon(\Delta) = \inf \frac{(L^\varepsilon y, y)}{(y, y)} \quad (y(x) = 0 \text{ for } x \notin \Delta),$$

where Δ is an n-dimensional cube, then for boundedness below (respectively, for discreteness of the spectrum) it is sufficient that $\lambda^\varepsilon(\Delta) > -C$ for all Δ of fixed size (respectively, that $\lambda^\varepsilon(\Delta) \to \infty$ when $\Delta \to \infty$, preserving its dimensions); for $m = 1$ one may put $\varepsilon = 0$. If $2m > n$, then a theorem analogous to Theorem 1 holds; the proof is carried out analogously and uses the theorem of M. Sh. Birman and B. S. Pavlov from ([3]).

Moscow State University
named after M. V. Lomonosov

Received
10 IV 1961

REFERENCES

1. A. M. Molchanov, *Tr. Mosk. matem. obshch.*, **2**, 169 (1953).
2. I. Brinck, *Mat. Scand.*, **7**, No. 1, 219 (1959).
3. M. Sh. Birman, B. S. Pavlov, *Vestn. Leningradsk. univ.*, ser. matem. i mekh., No. 1, issue 1, 61 (1961).

Note: Figure translations are in progress. See original paper for figures.