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1961

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Abstract

Full Text

MATHEMATICS

M. B. KAPILEVICH

**ON SINGULAR TRICOMI PROBLEMS IN
A NEIGHBORHOOD OF A FINITE AND
INFINITELY REMOTE SINGULAR LINE**

(Presented by Academician I. G. Petrovskii, 22 XI 1960)

Consider, in the domain G ($0 \leq x \leq y \leq x_0$) of the half-plane $y \geq x$, the equation

$$E(u, \beta, \beta') = (y - x)u_{xy} + \beta'u_x - \beta u_y = 0 \quad (1)$$

and call $u(x, y, \beta, \beta')$, $\bar{u}(x, y, \beta, \beta')$ solutions of the first and second singular Tricomi problems if, respectively:

$$u(x, x) = \tau(x), \quad u(0, y) = \tau(0) = 0; \quad \bar{u}_\eta(x, x) = \nu(x), \quad \bar{u}(0, y) = 0. \quad (2)$$

Suppose that $\tau(x)$ and $\nu(x)$ are twice continuously differentiable functions on the interval $(0, x_0)$, $\eta = -[(y - x)/(2 - a - a')]^{1-\alpha}$, $\alpha = \beta + \beta' < 1$, $a = 2\beta$, $a' = 2\beta'$. Introduce in (1) the variables $x = x$, $s = y - x$ and, denoting the new unknown function by $u(x, s)$, reduce (1), (2) to the boundary-value problems

$$F(u, \beta', \alpha) = s(u_{xs} - u_{ss}) + \beta'u_x - \alpha u_s = 0; \quad (3)$$

$$u(x, 0) = \tau(x), \quad u(0, s) = \tau(0) = 0; \quad \bar{u}_\eta(x, 0) = \nu(x), \quad \bar{u}(0, s) = 0. \quad (4)$$

For them the following Duhamel principle holds: if $U(x, s)$ and $\bar{U}(x, s)$ are integrals of equation (3) with boundary data $U(x, 0) = \bar{U}_\eta(x, 0) = 1$, $U(0, s) = \bar{U}(0, s) = 0$, then

$$u(x, s) = D_x \int_0^x U(x - \xi, s) \tau(\xi) d\xi = \int_0^x U(x - \xi, s) d\tau(\xi); \quad (5a)$$

$$\bar{u}(x, s) = D_x \int_0^x \bar{U}(x - \xi, s) \nu(\xi) d\xi = \int_0^x \bar{U}(x - \xi, s) d\nu(\xi). \quad (5b)$$

The discontinuous Duhamel kernels $U(x, s)$, $\bar{U}(x, s)$ reduce to modified incomplete Euler beta-functions $I_z(p, q) = B_z(p, q)/B_1(p, q)$, where $B_z(p, q) = p^{-1} z^p F(1 - q, p, p + 1; z)$, and have the form $U = I_t(\beta', 1 - \alpha) = 1 - I_{1-t}(1 - \alpha, \beta')$, $\omega^{-1} \bar{U} = I_t(1 - \beta, \alpha - 1) = 1 - I_{1-t}(\alpha - 1, 1 - \beta)$, $t = x/y$, $\omega = -[2(1 - \alpha)/s]^{\alpha-1} = 1/\eta$.

In the case of infinitely differentiable $\tau(x)$ and $\nu(x)$, the integrals (5a) and (5b), convergent only for $\beta' > -1$ and $\beta < 2$, may be replaced by the expansions

$$u = V(x, s, D_x) = \sum_{n=0}^{\infty} A_n (-s)^n B_t(\beta' + n, 1 - \alpha - n) D_x^n \tau(x) =$$

$$= \sum_{n=0}^{\infty} D_n \left(\frac{x}{s}\right)^{n+\beta'} \gamma^*(\beta' + n, x D_x) \tau(x), \quad A_n n! \Gamma(\beta') \Gamma(1 - \alpha) = \Gamma(1 - \beta),$$

$$D_n \Gamma(1 - \beta) = (-1)^n A_n \Gamma(\beta' + n) \Gamma(1 + n - \beta),$$

which in symbolic notation have the form

$$u = \frac{A_0}{\beta'} \left(\frac{x}{s}\right)^{\beta'} \Phi_1 \left(\beta', 1 - \beta, 1 + \beta', -\frac{x}{s}, -x D_x\right) \tau(x). \quad (6)$$

By analytically continuing the function Φ_1 , we arrive at analogous series also outside the domain $y > 2x$, where (6) converges. Replacing in (6) β , β' , $\tau(x)$ by $1 - \beta'$, $1 - \beta$, $\nu(x)$, respectively, we find, after multiplication by ω , the corresponding expansions for \bar{u} . The representation of the solutions u, \bar{u} in the forms (5), (6) reduces the discussion to the functions $I_z(p, q)$ studied in problems of mathematical statistics and tabulated ⁽¹⁾. In terms of these functions the Riemann functions are expressed also in the case of equation ⁽²⁾:

$$(y - x)u_{xy} + \beta(u_x - u_y) - b^2(y - x)u = 0. \quad (7)$$

Indeed, in order to construct $U(x, s)$, let us make in (7) the substitution

$$s = y - x, \quad t = x/y, \quad u = s^{-\beta} v,$$

after which, in the resulting equation,

$$s(1 - t)^2 v_{st} - t(1 - t)^2 v_{tt} - s^2 v_{ss} - (1 - t)^2 v_t + [\beta(\beta - 1) - b^2 s^2] v = 0$$

put

$$v = \sum_{n=0}^{\infty} s^{\beta+n} f_n(t).$$

Then we obtain the recurrent system of ordinary differential equations

$$t(1-t)^2 f_{m+2}''(t) - (1-t)[m + \beta + 1 + (m + \beta + 3)t] f_{m+2}'(t) + (m+2)(m+2\beta+1) f_{m+2}(t) + b^2 f_m(t) = 0, \quad (8)$$

where $m = -2, -1, 0, 1, 2, \dots$, $f_{-2}(t) = f_{-1}(t) \equiv 0$.

Moreover, in order to ensure satisfaction of the boundary conditions that determine $U(x, s)$, we shall assume

$$f_0(0) = 0, \quad f_0(1) = 1, \quad f_n(0) = 0, \quad \lim_{y \rightarrow x} \{s^n f_n(t)\} = 0 \quad (n = 1, 2, \dots).$$

Hence we first find

$$f_0(t) = I_t(\beta, 1 - \alpha),$$

i.e. the first term of the expansion

$$U = \sum_{n=0}^{\infty} s^n f_n(t)$$

coincides with the Riemann function for $E(u, \beta, \beta) = 0$.

Further analyzing the singularities of the solutions $f_{2n+1}(t)$ in neighborhoods of the points $t = 0$ and $t = 1$, we come to the conclusion that

$$f_{2n+1}(t) \equiv 0$$

for all $n = 0, 1, 2, \dots$. Finally differentiating (8) with respect to t , then, setting

$$f_{2n}'(t) = \Phi_{2n}(t),$$

we obtain

$$t(1-t)^2 \Phi_{2n}''(t) - (1-t)[2n + \beta - 2 + (2n + \beta + 4)t] \Phi_{2n}'(t) + 2[n(2n + 2\beta - 1) - 1 + (2n + \beta + 1)t] \Phi_{2n}(t) + b^2 \Phi_{2n-2}(t) = 0 \quad (n = 1, 2, \dots).$$

Since

$$\Phi_0(t) = c_0(\beta) t^{\beta-1} (1-t)^{-\alpha},$$

from these inhomogeneous differential equations we successively compute

$$\Phi_{2n}(t) = c_n(\beta) b^{2n} t^{n+\beta-1} (1-t)^{-2n-\alpha},$$

$$c_n(\beta) n! \Gamma(1 - \alpha) \Gamma(n + \beta) = \Gamma(1 - \beta),$$

and then find

$$f_{2n}(t) = c_n(\beta)b^{2n}B_t(n + \beta, 1 - 2n - \alpha).$$

Thus

$$U = \sum_{n=0}^{\infty} c_n(\beta)(bs)^{2n}B_t(n + \beta, 1 - 2n - \alpha),$$

and, analogously,

$$\bar{U} = \sum_{n=0}^{\infty} \bar{c}_n(bs)^{2n}B_t(1 + n - \beta, \alpha - 2n - 1),$$

where

$$\bar{c}_n = \omega c_n(1 - \beta).$$

These expressions for U and \bar{U} , after a preliminary quadratic transformation over B_t , reduce to confluent functions of Kampé de Fériet of the second order, of zero class, type 7 (see ⁽³⁾, p. 152). By the same method, in the case of (7), expansions of the form (6) are found as well. Namely, *in order to obtain the symbolic operator (6), it is enough to construct the Riemann function $V(x, s, \lambda)$ of the first singular Tricomi problem for the equation satisfied by the function*

$$v = \exp(-\lambda x)u \quad (\lambda = D_x = \text{const}).$$

Similar calculations, carried out here for the example (7), can also be performed for the more general Chaplygin equation, which plays an important role in applications. The limiting value

$$\lim_{s \rightarrow \infty} [s^{\beta'} u] = \frac{\Gamma(1 - \beta)}{\Gamma(1 - \alpha)} D_x^{-\beta'} \tau(x) = T(x),$$

found from (5), shows that

$$u(x, s)$$

is an integral of equation (3) that is regular in a neighborhood of the singular line $s = \infty$ with exponent equal to β' . Therefore, for the study of the function $u(x, s)$ as $s \rightarrow \infty$, it is convenient to introduce the variables

$$\sigma = 1/s, \quad w = s^{\beta'} u,$$

thereby reducing (3), (4) to the problem

$$w_{x\sigma} + \sigma^2 w_{\sigma\sigma} + (\beta' - \beta + 2)\sigma w_{\sigma} + \beta'(1 - \beta)w = 0, \quad (9)$$

$$w(x, 0) = T(x), \quad w(0, \sigma) = 0, \quad T(0) = 0. \quad (10)$$

Putting in (5a) $\tau(x) = \Gamma'(1 - \alpha)/\Gamma'(1 - \beta)D_x^{\beta'-1}T'(x)$, we obtain

$$w(x, \sigma, \beta, \beta') = D_x \int_0^x U(x - \xi, \sigma)T(\xi) d\xi = \int_0^x U_1'(x - \xi, \sigma) dT(\xi), \quad (11)$$

where $U(x, \sigma) = F(\beta', 1 - \beta, 1 - x\sigma)$. Conversely, (5a) arises if in (11) one passes from $T(x)$ to $\tau(x)$. Multiplying (9) by $2w_\sigma$, and then integrating the result over the rectangle $D = OAMBO$ with vertices $O(0, 0)$, $A(x_1, 0)$, $B(0, \sigma_1)$, $M(x_1, \sigma_1)$, then, assuming $w = 0$ on OA and OB , we arrive at the identity

$$\int_{BM} [\sigma^2 w_\sigma^2 + \beta'(1 - \beta)w^2] dx + \int_{AM} w_\sigma^2 d\sigma + 2(\beta' - \beta + 1) \iint_D \sigma w_\sigma^2 dx d\sigma = 0,$$

from which, for $\beta' > 0$, $\beta < 1$, it follows that $z \equiv 0$ in \bar{D} . The connection formulas make it possible to extend this uniqueness theorem also to the values $\beta' < 0$, $\beta > 1$.

Theorem 1. For $\beta'_1 > \beta'_2 > 0$, $\beta' = \beta'_1 - \beta'_2$, $\mu_1 \Gamma(\beta'_2) \Gamma(\beta') = \Gamma(\beta'_1)$, the solutions $w_k = w(x, \sigma, \beta, \beta'_k)$ ($k = 1, 2$) are related by the equality

$$w(x, \sigma, \beta, \beta'_2) = \mu_1 \int_0^1 \xi^{\beta'_2 - 1} (1 - \xi)^{\beta' - 1} w(x, \xi\sigma, \beta, \beta'_1) d\xi, \quad (12)$$

to which, in the case $T(x) \in C_{n+1}$ ($0 \leq x \leq x_0$), there corresponds the expansion

$$w(x, \sigma, \beta, \beta'_2) = \sum_{k=0}^n \frac{(\beta')_k}{(\beta'_1)_k} k! (-\sigma)^k D_\sigma^k w(x, \sigma, \beta, \beta'_1) + R_n, \quad (13)$$

$$R_n = \frac{(-1)^{n+1} \Gamma(\beta'_1)}{\Gamma(\beta') \Gamma(\beta'_2 + n + 1)} \times \int_0^1 \xi^{\beta'_2 + n} F(1 - \beta', \beta'_2, \beta'_2 + n + 1, \xi) D_\xi^{n+1} w(x, \xi\sigma, \beta, \beta'_1) d\xi.$$

Theorem 2. For $0 \leq \beta_1 < \beta_2 < 1$, $\beta = \beta_2 - \beta_1$, $\mu_2 \Gamma(1 - \beta_2) \Gamma(\beta) = \Gamma(1 - \beta_1)$, the relation holds

$$w(x, \sigma, \beta_2, \beta') = \mu_2 \int_0^1 \xi^{-\beta_2} (1 - \xi)^{\beta - 1} w(x, \xi\sigma, \beta_1, \beta') d\xi. \quad (14)$$

If $T(x)$ is infinitely differentiable on the interval $(0, x_0)$, and $|T^{(n)}(x)| \leq M$ ($n = 0, 1, 2, \dots$), then (14) generates, in the limit, the expansion

$$w(x, \sigma, \beta, \beta'_2) = {}_1F_1(\beta', \beta'_1; -\delta_\sigma) w(x, \sigma, \beta, \beta'_1) \quad (\delta_\sigma = \sigma D_\sigma), \quad (15)$$

which for $\beta' = -m$ ($m = 1, 2, \dots$) reduces to the finite recurrence relations

$$(\beta')_m w(\beta, \beta' + m) = m! L_m^{\beta' - 1}(-\delta_\sigma) w(\beta, \beta') = \sigma^{1 - \beta'} D_\sigma^m [\sigma^{\beta' + m - 1} w(\beta, \beta')].$$

It also follows from (15) that

$$w(\beta, 1) = \Gamma(\beta') \gamma^*(\beta' - 1, \delta_\sigma) w(\beta, \beta')$$

for arbitrary $\beta' \neq 0, -1, -2, \dots$. Analogous expansions also arise from (14). For example, for adjacent functions $w(\beta - m, \beta')$ ($m = 0, 1, 2, \dots$), we find in this way

$$(1 - \beta)_m w(\beta - m, \beta') = \sigma^\beta D_\sigma^m [\sigma^{m-\beta} w(\beta, \beta')].$$

Solving the integral equations (12), (14), we obtain the inverse transformation operators T_σ^{-1} :

$$\Gamma(\beta'_1) w(\beta, \beta'_1) = \Gamma(\beta'_2) \sigma^{1-\beta_2} D_\sigma D_\sigma^{\beta'_2-1} [\sigma^{\beta_1-1} w(\beta, \beta'_2)] \quad (\beta'_1 > 0, \beta' < 1),$$

$$\Gamma(1 - \beta_1) w(\beta_1, \beta') = \Gamma(1 - \beta_2) \sigma^2 D_\sigma D_\sigma^{\beta_2-1} [\sigma^{-\beta} w(\beta_2, \beta')] \quad (0 < \beta < 1).$$

It is also interesting to compare (9) with its confluent case

$$z_{x\sigma} + a\sigma z_\sigma + a\beta' z = 0 \quad (a > 0),$$

which arises from (9) if, after replacing x, β by $\varepsilon x, -a/\varepsilon$, one puts there $\varepsilon = 0$, and, in particular, to compare (9) with the telegraph equation

$$v_{x\sigma} + c^2 v = 0,$$

which approximates (9) for $c^2 = \beta'(1 - \beta)$ in a neighborhood of the line $\sigma = 0$.

Theorem 3. If $\beta < 1$, $\beta' > 0$, $c > 0$, $\mu \Gamma(\beta') \Gamma(1 - \beta) = 2c^{1+\beta'-\beta}$, then the solutions $w(x, \sigma, \beta, \beta')$, $z(x, \sigma, \beta')$, $v(x, \sigma, c)$ of problem (10) are transformed into one another by the relations

$$w(x, \sigma, \beta, \beta') = \frac{a^{1-\beta}}{\Gamma(1 - \beta)} \int_0^\infty \xi^{-\beta} e^{-a\xi} z(x, \xi\sigma, \beta') d\xi,$$

$$w(x, \sigma, \beta, \beta') = \mu \int_0^\infty \xi^{(\beta'-\beta-1)/2} K_{\alpha-1}(2c\sqrt{\xi}) v(x, \xi\sigma, c) d\xi.$$

Inverting these equalities makes it possible to obtain the known inversion formulas for the Laplace and Hankel transforms.

For the integrals w, z, v one can also effectively construct transformation operators T_x , for example

$$\begin{aligned} w(\beta, \beta'_2) &= D_x \int_0^x Q_1(\xi) u(\xi, \sigma, \beta'_1) d\xi = \\ &= D_x \int_0^x Q_2(\xi) v(\xi, \sigma, c) d\xi, \end{aligned}$$

if $T_2(x) = P(x)T_1(x)$, while Q_1 and Q_2 are defined by the expressions

$$Q_1 = -e^{a\xi\sigma} D_\xi e^{-a\xi\sigma} \times \\ \times \int_\xi^x P(t) {}_1F_1[1 - \beta'_1, 1, -a\sigma(t - \xi)] F[\beta'_2, 1 - \beta, 1, -\sigma(x - t)] dt, \\ Q_2 = -D_\xi \int_\xi^x P(t) I_0[2c\sqrt{\sigma(t - \xi)}] F[\beta'_2, 1 - \beta, 1, -\sigma(x - t)] dt.$$

These equalities correspond to the expansions

$$w(\beta, \beta'_2) = {}_2F_0[\beta'_2, 1 - \beta, -\sigma D_x^{-1}][P(x)T_1(x)],$$

where

$$T_1(x) = \exp(-ax\sigma)(1 - a\sigma D_x^{-1})^{-\beta'_1} [\exp(ax\sigma) \times u(x, \sigma, \beta'_1)] = \exp(c^2\sigma D_x^{-1})v(x, \sigma, c),$$

and the divergent series ${}_2F_0(a, b, -z^{-1}) = z^a \Psi(a, a - b + 1, z)$ in integral cases reduce to known Bessel polynomials⁽⁴⁾. Products of the operators T_x and T_σ give a series of iterated relations; thus, for $\beta' = \beta'_1 - \beta'_2$, $\lambda > 0$, $\beta'_2 > 0$, $\beta'_1 > \beta'_2 + \lambda$, $\mu\Gamma(\beta'_2)\Gamma(\lambda)\Gamma(\beta' - \lambda) = \Gamma(\beta'_1)$, we find

$$w(\beta, \beta'_2) = \mu \int_0^1 \int_0^1 t^{\frac{1}{2}\beta' - 1} (1 - t)^{\beta' - \lambda - 1} (1 - \xi)^{\lambda - 1} \times \\ \times F(\beta'_1 - \lambda - 1, -\lambda, \beta' - \lambda, 1 - t) w(x, \xi\sigma t, \beta, \beta'_1) d\xi dt.$$

With the aid of connection formulas, a large number of relations (integral ones and in the form of infinite series) are established for the hypergeometric functions of Appell, Horn, Humbert, Kampé de Fériet of the first and higher orders with two independent variables, as well as for Lauricella functions and their confluent forms, to which $u, \bar{u}, w, \bar{w}, z, \bar{z}, v, \bar{v}$ reduce under a suitable choice of the values of $\tau(x), \gamma(x), T(x), P(x)$.

Moscow Evening
Metallurgical Institute

Received
19 XI 1960

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