AN ITERATIVE METHOD FOR SOLVING A LINEAR PROGRAMMING PROBLEM

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Abstract

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AN ITERATIVE METHOD FOR SOLVING A LINEAR PROGRAMMING PROBLEM

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Let R_n be an n-dimensional Euclidean space consisting of vectors $h = (h^{(1)}, h^{(2)}, \ldots, h^{(n)})$. We formulate the linear programming problem as follows. A system of linear inequalities is given

$$\sum_{j=1}^{n} a_j^{(i)} h^{(j)} \geqslant p^{(i)}, \qquad i = 1, 2, \ldots, m;$$

$$h^{(j)} \geqslant 0, \quad j = 1, 2, \ldots, n, \tag{1}$$

where $a_j^{(i)}, p^{(i)}$ are given numbers. In addition, a vector $c \in R_n$ is given. The system (1) defines a closed convex set $Q \subset R_n$. It is required to find a vector $h_0 \in Q$ such that $(c, h_0) \leqslant (c, h)$ for all $h \in Q$. We shall call this problem problem I.

In addition to it, we pose another problem II: find a vector $h_\sigma \in Q$ such that $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h)$ for all $h \in Q$. Here B is a given matrix and $\sigma > 0$ is a number.

The proposed method for solving problem I consists in replacing it by problem II for some small σ and a specially chosen matrix B. To solve the latter problem an iterative process is constructed.

Lemma. If Q' is a nonempty, bounded, convex, closed set in R_n, and F is a continuous operation from Q' into R_n, then there exists a vector $h' \in Q'$ such that $(Fh', h') \leqslant (Fh', h)$ for all $h \in Q'$.

The proof of this assertion can be obtained by slightly modifying the arguments in Theorem 3 of ([3]).

Theorem 1. If there exists a vector h_0 solving problem I, and the matrix B is positive definite: $(Bh, h) \geqslant \gamma \|h\|^2$, then for any $\sigma > 0$ there exists a vector h_σ solving problem II, and moreover

$$\|h_\sigma\| \leqslant \frac{\|B\|}{\gamma} \|h_0\|.$$

Proof. Adjoin to (1) the inequality

$$\sum_{j=1}^{n} h^{(j)} \leqslant \frac{\sqrt{n}}{\gamma}\|B\| \cdot \|h_0\| + 1.$$

The system of inequalities thus obtained defines a domain Q'. By the lemma, there is a vector $h_\sigma \in Q'$ such that $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h)$ for all $h \in Q'$. Since $h_0 \in Q'$, we have $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h_0)$. Taking into account that $(c, h_0) \leqslant (c, h_\sigma)$, we obtain $\sigma\gamma\|h_\sigma\|^2 \leqslant \sigma(Bh_\sigma, h_\sigma) \leqslant \sigma(Bh_\sigma, h_0) \leqslant \sigma\|B\|\,\|h_\sigma\|\,\|h_0\|$,

$$\|h_\sigma\| \leqslant \frac{\|B\|}{\gamma}\|h_0\|. \tag{2}$$

We shall show that h_σ solves problem II. If $h \in Q'$, then, by the construction of h_σ, we have $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h)$. Let $h \in Q \setminus Q'$. From (2) we obtain

$$\sum_j h_\sigma^{(j)} \leqslant \sqrt{n}\,\|h_\sigma\| \leqslant \sqrt{n}\,\frac{\|B\|}{\gamma}\,\|h_0\|.$$

Consequently,

$$\Delta = \sum_i h^{(i)} - \sum_j h_\sigma^{(j)} > 1.$$

Put $\lambda = 1 : \Delta$ and $h' = \lambda h + (1-\lambda) h_\sigma$. It is not hard to verify that $h' \in Q'$, and consequently $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h')$. Substituting in place of h' its expression, we obtain $(c + \sigma B h_\sigma, h_\sigma) \leqslant (c + \sigma B h_\sigma, h)$. This proves the theorem.

Corollary. Since the vector h_σ solves the problem of minimizing the functional $L(h) = (c + \sigma B h_\sigma, h)$, it follows from $(1, 2)$, under the conditions of Theorem 1, that there exists a vector $v_\sigma = (v_\sigma^{(1)}, v_\sigma^{(2)}, \dots, v_\sigma^{(m)})$ such that

$$\sum_{i=1}^{m} a_j^{(i)} v_\sigma^{(i)} \leqslant c^{(j)} + \sigma(Bh_\sigma)^{(j)}, \quad \text{if } h_\sigma^{(j)} = 0;$$

$$\sum_{i=1}^{m} a_j^{(i)} v_\sigma^{(i)} = c^{(j)} + \sigma(Bh_\sigma)^{(j)}, \quad \text{if } h_\sigma^{(j)} > 0; \tag{3}$$

$$v_\sigma^{(i)} \geqslant 0, \quad i = 1, 2, \dots, m;$$

$$v_\sigma^{(i)} = 0, \quad \text{if } \sum_{j=1}^{n} a_j^{(i)} h_\sigma^{(j)} > p^{(i)}.$$

Theorem 2. The following assertions are valid:
a) the solution of problem II is unique;
b) for $\sigma > 0$, h_σ depends continuously on σ;
c) if $\sigma < \sigma'$, then $(c, h_\sigma) \leqslant (c, h_{\sigma'})$;
d) there exists $\sigma_0 > 0$ such that for $\sigma \leqslant \sigma_0$ the vector h_σ is a solution of problem I.

We omit the proof.

Denote $a_j = (a_j^{(1)}, a_j^{(2)}, \dots, a_j^{(m)})$, $p = (p^{(1)}, p^{(2)}, \dots, p^{(m)})$, and assume that all $a_j \neq 0$, $j = 1, 2, \dots, n$. As B take the triangular matrix in which zeros stand above the main diagonal, and the (μ, ν)-th element $(\nu \leqslant \mu)$ is equal to (a_μ, a_ν). For this matrix $(Bh, h) \geqslant \gamma \|h\|^2$, where $2\gamma = \min \|a_j\|^2$.

From the first two relations (3) we find

$$h_\sigma^{(j)} = \begin{cases} \dfrac{(v_{\sigma,j-1}, a_j) - c^{(j)}}{\sigma \|a_j\|^2}, & \text{if } (v_{\sigma,j-1}, a_j) \geqslant c^{(j)}, \\ 0, & \text{if } (v_{\sigma,j-1}, a_j) < c^{(j)}, \end{cases} \tag{4}$$

where

$$v_{\sigma,j-1} = v_\sigma - \sigma \sum_{\nu=1}^{j-1} h_\sigma^{(\nu)} a_\nu.$$

Put $v'_\sigma = v_{\sigma,n} + \sigma p$. From (1) and the last two relations (3) we conclude that

$$v_\sigma^{(i)} = \begin{cases} v_\sigma a'^{(i)}, & \text{if } v_\sigma a'^{(i)} \geqslant 0; \\ 0, & \text{if } v_\sigma a'^{(i)} < 0. \end{cases}$$

We now construct the following iterative process:

$$\text{I.} \qquad v_{k,0} = v_k.$$

$$\text{II.} \qquad h_k^{(j)} = \begin{cases} \dfrac{(v_{k,j-1}, a_j) - c^{(j)}}{\sigma \|a_j\|^2}, & \text{if } (v_{k,j-1}, a_j) \geqslant c^{(j)}, \\ 0, & \text{if } (v_{k,j-1}, a_j) < c^{(j)}; \end{cases} \tag{4'}$$

$$v_{k,j} = v_{k,j-1} - \sigma h_k^{(j)} a_j.$$

III.

$$v_k' = v_{k,n} + \sigma p.$$

IV.

$$v_{k+1}^{(i)} = \begin{cases} {v'}_k^{(i)}, & \text{if } {v'}_k^{(i)} \geq 0, \\ 0, & \text{if } {v'}_k^{(i)} < 0. \end{cases}$$

V.

$$\left(v_{k+1}^{(1)}, v_{k+1}^{(2)}, \dots, v_{k+1}^{(m)}\right) = v_{k+1}, \qquad \left(h_k^{(1)}, h_k^{(2)}, \dots, h_k^{(n)}\right) = h_k.$$

Put

$$\Delta_\sigma^{(j)} = \sigma \|a_j\|^2 h_\sigma^{(j)} - (v_{\sigma,j-1}, a_j) + c^{(j)}; \qquad \Delta_k^{(j)} = \sigma \|a_j\|^2 h_k^{(j)} - (v_{k,j-1}, a) + c^{(j)}.$$

From (4) and (4′) we conclude that
$\Delta_\sigma^{(j)} \geq 0$, $\Delta_k^{(j)} \geq 0$,
$h_\sigma^{(j)} \Delta_\sigma^{(j)} = h_k^{(j)} \Delta_k^{(j)} = 0$. Consequently,

$$\left(h_k^{(j)} - h_\sigma^{(j)}\right)\left(\Delta_k^{(j)} - \Delta_\sigma^{(j)}\right) = -\left(\Delta_k^{(j)} h_\sigma^{(j)} + \Delta_\sigma^{(j)} h_k^{(j)}\right) \leq 0.$$

Next,

$$\begin{aligned} \|v_{k,j} - v_{\sigma,j}\|^2 &= \|v_{k,j-1} - v_{\sigma,j-1} - \sigma h_k^{(j)} a_j + \sigma h_\sigma^{(j)} a_j\|^2 \\ &= \|v_{k,j-1} - v_{\sigma,j-1}\|^2 - \sigma^2 \|a_j\|^2 \left(h_k^{(j)} - h_\sigma^{(j)}\right)^2 \\ &\quad + 2\sigma \left(h_k^{(j)} - h_\sigma^{(j)}\right)\left(\Delta_k^{(j)} - \Delta_\sigma^{(j)}\right). \end{aligned}$$

Denoting $\tau = \sigma^2 \min \|a_j\|^2$, we obtain

$$\|v_{k,j} - v_{\sigma,j}\|^2 \leq \|v_{k,j-1} - v_{\sigma,j-1}\|^2 - \tau \left(h_k^{(j)} - h_\sigma^{(j)}\right)^2.$$

Adding these inequalities for $j = 1, 2, \dots, n$, we obtain

$$\|v_{k,n} - v_{\sigma,n}\|^2 \leq \|v_{k,0} - v_{\sigma,0}\|^2 - \tau \|h_k - h_\sigma\|^2.$$

Next,

$$\|v_{k+1}-v_\sigma\|^2 \le \|v'_k-v'_\sigma\|^2 = \|v_{k,n}-v_{\sigma,n}\|^2 \le \|v_{k,0}-v_{\sigma,0}\|^2-\tau\|h_k-h_\sigma\|^2 = \|v_k-v_\sigma\|^2-\tau\|h_k-h_\sigma\|^2.$$

Applying the obtained inequality for smaller k, we obtain

$$\|v_{k+1} - v_\sigma\|^2 \le \|v_0 - v_\sigma\|^2 - \tau \sum_{\nu=0}^{k} \|h_\nu - h_\sigma\|^2.$$

Since $\tau > 0$, it follows from this that $h_\nu \to h_\sigma$ as $\nu \to \infty$.

Thus, we have shown that the proposed iterative process gives, in the limit, the solution of problem II, which in turn approximates problem I, as follows from Theorems 1 and 2. It can be shown that the sequence $\{v_k\}$ also converges to some vector v_σ satisfying a system of inequalities analogous to (3).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.