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Abstract

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M. Z. SOLOMYAK

ON ELLIPTIC OPERATORS ON TWO-DIMENSIONAL MANIFOLDS

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1. Let D be a plane domain or a sufficiently smooth orientable closed surface in three-dimensional Euclidean space. In D consider the elliptic operator

$$Au = \sum_{l=0}^{s} A_l(M) \frac{\partial^s u}{\partial x^l \partial y^{s-l}} + Tu, \tag{1}$$

where u is a vector with p components; $A_l(M)$ are square $p \times p$ matrices; their elements are assumed to be complex functions, continuously dependent on the point $M \in D$. The symbol Tu denotes the aggregate of lower-order terms. In the spatial case differentiation is carried out with respect to local orthogonal coordinates on the surface. Ellipticity of the operator (1) means that for $M \in D$, for all real λ, the condition

$$\left| \det \left[\sum_{l=0}^{s} A_l(M) h_x^l h_y^{s-l} \lambda^{s-l} \right] \right| \geq c_0 (1 + \lambda^2)^{sp/2}, \tag{2}$$

is satisfied, where h_x, h_y are the scale factors along the axes x, y at the point M.

On the left-hand side of relation (2) there stands a polynomial of degree $m = sp$ of the form

$$P_m(M, \lambda) = \sum_{l=0}^{m} a_l(M) h_x^l h_y^{m-l} \lambda^{m-l}. \tag{3}$$

Upon passing to another orthogonal system (ξ, η), making an angle α with the former at the point M, the polynomial (3) takes the form

$$P'_m(M, \lambda) = \sum_{l=0}^{m} a_l(M) \left[h_\xi \cos\alpha - \lambda h_\eta \sin\alpha \right]^l \left[h_\xi \sin\alpha + \lambda h_\eta \cos\alpha \right]^{m-l}. \tag{4}$$

Polynomials (3), transforming according to formula (4) and satisfying a condition of the form (2), will be called **elliptic**; the set of all elliptic polynomials of order m on D will be denoted by $E_m(D)$. By the symbol $E_s^p(D)$ we shall denote the set of all elliptic operators of the form (1).

In the present note the algebraic structure of the polynomials from $E_m(D)$ is investigated, and the structure of the connected components in the set $E_s^1(D)$ is also clarified.

Theorem 1. Let D be a sufficiently smooth orientable closed surface not homeomorphic to a torus. Then every elliptic poly-

on D has even order, and its roots are equally distributed between the upper and lower half-planes of the complex plane.

For equations with many independent variables, an analogous theorem was proved by Ya. B. Lopatinskii (^{1}); for systems of class E_p^r on the two-dimensional sphere this result, without proof, is given in the paper of A. I. Vol' pert (^{2}). In the plane (and also on the torus), as is known, there exist elliptic polynomials not satisfying this condition, for example the polynomial $P(\lambda) = \lambda - i$, corresponding to the Cauchy–Riemann operator.

Theorem 2. *Let the surface D satisfy the hypotheses of Theorem 1. Then the set of connected components in $E_s^1(D)$ is empty for odd s, and for even s is isomorphic to the group of ∇^1-homologies of the surface D over the field of residues modulo two.*

Theorem 3. *Let D be a plane domain or a surface homeomorphic to a torus. Then the set $E_s^1(D)$, for arbitrary s, splits into classes determined by the number of roots of the characteristic polynomial lying in the upper half-plane. Each class, in turn, splits into connected components, whose set is isomorphic to the group $\nabla^1(D, J_0)$.*

2. We outline the proof. Let $\dfrac{h_x}{h_y}\lambda_l(M)$ be the roots of the polynomial (3). Consider the one-parameter family of polynomials

$$P_m(M, \lambda; t) = a_m(M; t) \prod_{l=1}^{m} \left|\lambda h_y - \lambda_l(M; t) h_x\right|, \qquad 0 \leq t \leq 1, \tag{5}$$

where

$$\frac{h_x}{h_y}\lambda_l(M; t) = \frac{(2-t) h_x \lambda_l(M) \pm i t h_y}{\pm \dfrac{h_x}{i} t \lambda_l(M) + (2-t) h_y}. \tag{6}$$

The sign in (6) is taken to coincide with the sign of $\operatorname{Im} \lambda_l(M)$. Obviously,

$$\lambda_l(M;0) = \lambda_l(M); \qquad \frac{h_x}{h_y}\lambda_l(M;1) = \pm i.$$

Formula (6) is invariant under an orthogonal transformation of coordinates.

Using this circumstance, it is not difficult to choose the coefficient $a_m(M;t)$ so that the following conditions are satisfied: $a_m(M;t)$ is continuous and does not vanish on $D \times [0,1]$; $a_m(M;0)$ coincides with the leading coefficient of the polynomial (3); $|a_m(M;1)| \equiv 1$. These conditions can be satisfied independently of whether or not the surface D admits a single coordinate net without singular points. It is easy to see that the polynomials (5) remain elliptic for all $t \in [0,1]$. Thus it has been established that the polynomial (3) can be joined simultaneously on all of D, without destroying ellipticity and continuity of the coefficients, by a parameter to a polynomial of the form

$$Q_m(M;\lambda) = e^{i\theta(M)}(\lambda h_y - ih_x)^k(\lambda h_y + ih_x)^{m-k}, \tag{7}$$

where k is the number of roots of the polynomial (3) in the upper half-plane.

Applying formula (4), we find that, under a rotation of the coordinate system through an angle α, the argument $\theta(M)$ in (7) acquires an increment equal to $(2k-m)\alpha$. It follows easily from this that, in the case $m \neq 2k$, the directions making with the x-axis the angles

$$\varphi_n(M) = \frac{\theta(M) + 2\pi n}{m-2k} \quad (n = 0, \dots, m-2k-1)$$

are invariant under coordinate transformations. Thus on D there is defined an $|m-2k|$-valued field of tangent directions, depending continuously on the point $M \in D$. Covering D by a many-sheeted surface on which the field is single-valued, and applying the theorem on tangent vector fields on many-forms, we arrive at the conclusion that on a surface not homeomorphic to a torus it is necessary that $m = 2k$. This proves Theorem 1.

Let us further note that for $m = 2k$ the argument $\theta(M)$ does not depend on the choice of the coordinate system. Consequently, on surfaces not homeomorphic to a torus, two elliptic polynomials of order m belong to the same connected component if and only if the corresponding mappings $M \to \theta(M)$ are homotopic. The mapping $M \to \theta(M)$ assigns to each one-dimensional cycle on D an integer equal to the increment of $\theta(M)$ upon traversing the cycle, divided by 2π; moreover, any cycle homologous to zero corresponds to zero increment. Each class of mutually homotopic mappings $M \to \theta(M)$ generates a homomorphism

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[*] Concerning the topological terminology, see, for example, (3).

of the group $\Delta^1(D, J_0)$ into the additive group of integers; the set of all such homomorphisms is the group $\nabla^1(D, J_0)$. To prove Theorem 2 it remains to note that the structure of the set of operators $E^1_s(D)$ and of the set of polynomials $E_s(D)$ is the same.

Under the hypotheses of Theorem 3, any distribution of roots between the half-planes is possible; the behavior of the argument $\theta(M)$ may be studied in any coordinate system prescribed on all of D. After these remarks the proof of Theorem 3 is evident.

3. One equation of elliptic type with complex coefficients is equivalent to an elliptic system of two equations with real coefficients. On the other hand, every elliptic system of two equations with real coefficients can, by continuation with respect to a parameter, be transformed into a system reducible to a single equation. Thus, Theorems 2 and 3 also characterize the structure of the set of all elliptic systems of two equations with real coefficients. For systems in simply connected plane domains, a homotopic classification was carried out by B. V. Boyarskii (^{4}).

4. Following I. M. Gelfand (^{5}), we include two operators in one homotopy class if they can be joined by a parameter without destroying the continuity of the coefficients and the property of ellipticity. The author believes that, in the case when the group $\nabla^1(D, J_0)$ is nontrivial, such a definition is not entirely natural. For example, in the annulus $r_1^2 \leq x^2 + y^2 \leq r_2^2$, the operators Δu and $e^{i\varphi}\Delta u$ (φ is the polar angle of the point $M(x, y)$) belong to different components, whereas, obviously, the solvability properties for equations with these operators are the same.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.